Name ________________________

Class ________________

2021

HSC Assessment 1 Term 4

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 45 minutes
- Board approved calculators may be used.
- Write using black or blue pen
- All necessary working should be shown in all questions
- Write your student number at the top of every page of your answer sheets.
- NESA Reference Sheet is provided

Total marks – 35

Attempt all questions

Section A – Answer on the Multiple Choice Answer Sheet

Section B - Start each question on a new sheet of paper.

NESA Reference Sheet is provided

This paper MUST NOT be removed from the examination room

Section A Multiple Choice (5 Marks)

QUESTION 1

How many values of k are there such that $(1-k^2)x^3 + kx^2 + 2x + 1$ is a monic polynomial?

A. 1
B. 2
C. 3
D. 4

QUESTION 2

How many solutions are there for the equation $\sin x \left(\tan\frac{x}{2} - 1\right) = 0$ for $x \in [-\pi, \pi\,]$?

A. 2

B. 3

C. 4

D. 5

QUESTION 3

The quartic polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are real constants has a double root at the origin. Which of the following options can NOT be equal to zero?

A. a

B. b

C. c

D. d

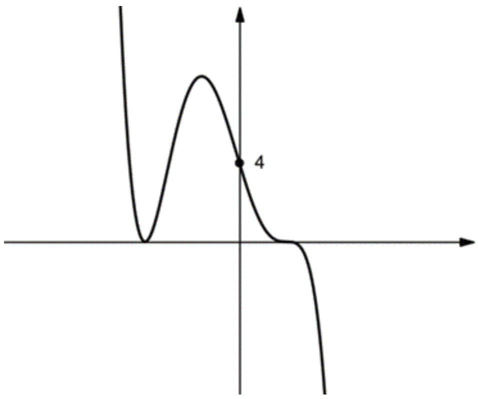
QUESTION 4

If $t = \tan\frac{\theta}{2}$, then $\frac{\sec\theta}{\sin\theta}$ in terms of t is:

A. $\frac{1+2t^2+t^4}{2t-2t^3}$
B. $\frac{1+t^2}{2t}$
C. $\frac{1-t^2}{2t}$
D. $\frac{1+2t^2+t^4}{4t^2}$

QUESTION 5

Which of the following options represents the graph shown below?



A. $y = (1 - x)^2(x + 2)^3$

B. $y = (1 - x)^3(x + 2)^2$

C. $y = -(1 - x)^2(x + 2)^3$

D. $y = -(1 - x)^3(x + 2)^2$

SECTION B

QUESTION 6 Start a new page (3 marks) **Marks**

Use Mathematical Induction to prove that 3

$$1^2 \times 2 + 2^2 \times 2^2 + 3^2 \times 2^3 + \cdots + n^2 \times 2^n = (n^2 - 2n + 3)2^{n+1} - 6$$

for all positive integers n ≥ 1.

QUESTION 7 Start a new page (4 marks)

i) Rewrite the expression $3\sin\theta - 2\cos\theta$ as $R\sin(\theta + \alpha)$ where $R > 0$ and $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. 2
Show all necessary working. You may express α correct to 2 decimal places.

ii) Hence, using your answer in i), find the maximum value of the expression $-3\sin\theta + 2\cos\theta - 1$. 2

QUESTION 8 Start a new page (5 marks)

a) Given that $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$
solve $\tan^3 x - 3\tan^2 x - 3\tan x + 1 = 0$ for $x \in [0, 2\pi]$ 3

b) Let α, β and γ be the roots of the equation $x^3 + 5x^2 - 2x + 7 = 0$. 2
Find the value of
$$\frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta}$$

QUESTION 9 Start a new page (5 marks)

a) By letting $t = \tan\frac{\theta}{2}$, solve the equation $\cos\theta + 3\sin\theta = -1$ for $\theta \in [0, 2\pi]$. 3
Express your answer in exact values or correct to 2 decimal places where appropriate.

b) Prove that if the polynomials $f(x)$ and $g(x)$ have a common root of α, then $(x - \alpha)$ is a factor of $f(x) - g(x)$. 2

QUESTION 10 Start a new page (5 marks)

The roots of the quadratic equation $x^2 - 3x + 4 = 0$ are α and β.

i) Show that $\alpha^3 = 5\alpha - 12$. 2

ii) By considering a similar expression in terms of β, find the value of $\alpha^3 + \beta^3$. 3

QUESTION 11 Start a new page (8 marks)

a) Solve $\sin\left(\theta - \frac{\pi}{3}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2}$ for $\theta \in [0, 2\pi]$ 2

b) In this question, you are given that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

i) Hence, show that $x = \sin\frac{\pi}{10}$ and $x = \sin\frac{13\pi}{10}$ are solutions of the equation $16x^5 - 20x^3 + 5x - 1 = 0$. 2

ii) Given that $(x-1)(4x^2 + ax - 1)^2 = 16x^5 - 20x^3 + 5x - 1$ find a and hence show that 4

$$\sin\frac{\pi}{10} - \sin\frac{3\pi}{10} = -\frac{1}{2}.$$

End of examination

2021 HSC Ext 1 Task 1

MATHEMATICS Extension 1 : Question MC		
Suggested Solutions	**Marks**	**Marker's Comments**
1. B $(1-k^2)x^3 + kx^2 + 2x + 1$ monic: $1-k^2 = 1$ $k^2 = 0$ $\therefore k = 0$ OR $1-k^2 = 0$ $k^2 = 1$ $k = \pm 1$ since coefficient of x^2 is k $\therefore k = 1$ $\therefore$ 2 solutions $k = 0, 1$ 2. A $\sin x(\tan\frac{x}{2} - 1) = 0 \quad x \in [-\pi, \pi]$ $\therefore \sin x = 0$ $x = -\pi, 0, \pi$ $\tan\frac{x}{2} - 1 = 0$ $\tan\frac{x}{2} = 1$ $\frac{x}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\therefore \frac{x}{2} = \frac{\pi}{4}$ $x = \frac{\pi}{2}$ <u>But</u> $\tan\frac{-\pi}{2}$ and $\tan\frac{\pi}{2}$ have no solutions $\therefore x = 0, \frac{\pi}{2}$ 3. B for double root x^2 must exist $\therefore$ b cannot equal 0.		

Suggested Solutions	Marks	Marker's Comments
4. A $t = \tan\frac{\theta}{2}$ $\dfrac{\sec\theta}{\sin\theta} = \dfrac{\frac{1+t^2}{1-t^2}}{\frac{2t}{1+t^2}}$ $= \dfrac{(1+t^2)^2}{2t(1-t^2)}$ $= \dfrac{1+2t^2+t^4}{2t-2t^3}$ 5. B triple root at $x=1$ ie $(x-1)^3$ double root at $x=-2$ ie $(x+2)^2$ graph starts in the 4th quadrant ie leading coefficient is negative. $\therefore y = -(x-1)^3(x+2)^2$ $\therefore y = (1-x)^3(x+2)^2$ check that at $x=0$, $y=4$ $y = (1-0)^3(0+2)^2$ $= 4$		

Mathematics Extension 1
HSC Assessment 1, T4

Question 6

Question 6

Use Mathematical Induction to prove that

$1^2 \times 2 + 2^2 \times 2^2 + 3^2 \times 2^3 + \cdots + n^2 \times 2^n = (n^2 - 2n + 3)2^{n+1} - 6$ for all positive integers $n \geq 1$.

Answer

Step 1: verify for n=1

$1^2 \times 2 + 2^2 \times 2^2 + 3^2 \times 2^3 + \cdots + n^2 \times 2^n = (n^2 - 2n + 3)2^{n+1} - 6$ for all positive integers $n \geq 1$.

$$\begin{aligned} \text{LHS} &= 1^2 \times 2 \\ &= 2 \\ \text{RHS} &= (1^2 - 2 \times 1 + 3)2^{1+1} - 6 \\ &= 8 - 6 \\ &= 2 \\ &= \text{LHS} \end{aligned}$$

Hence, verified for n=1

1

Answer (cont.)

Step 2: assume the given statement is true for some integer $k \geq 1$.

$$1^2 \times 2 + 2^2 \times 2^2 + 3^2 \times 2^3 + \cdots + k^2 \times 2^k = (k^2 - 2k + 3)2^{k+1} - 6$$

Step 3: verify for $n = k + 1$

$1^2 \times 2 + 2^2 \times 2^2 + 3^2 \times 2^3 + \cdots + k^2 \times 2^k + (k+1)^2 \times 2^{k+1}$

$= (k^2 - 2k + 3)2^{k+1} - 6 + (k+1)^2 \times 2^{k+1}$ (using assumption) 1

$= 2^{k+1}\left(k^2 - 2k + 3 + (k+1)^2\right) - 6$

$= 2^{k+1}\left(k^2 - 2k + 3 + k^2 + 2k + 1\right) - 6$

$= 2^{k+1}\left(2k^2 + 4\right) - 6$

$= 2^{k+1} \times 2\left(k^2 + 2\right) - 6$

$= 2^{k+2}\left(k^2 + 2\right) - 6$

$= \left((k+1)^2 - 2(k+1) + 3\right) \times 2^{k+2} - 6$

Hence verified for $n = k + 1$ 1

$\left((k+1)^2 - 2(k+1) + 3\right) \times 2^{k+2} - 6$
$= \left(k^2 + 2k + 1 - 2k - 2 + 3\right) \times 2^{k+2} - 6$
$= (k^2 + 2) \times 2^{k+2} - 6$

Answer (cont.)

By the principle of mathematical induction, the given statement is true for any positive integer $n \geq 1$.

Question 7

Monday, 13 December 2021 10:18 pm

a) $3\sin\theta - 2\cos\theta = R\sin(\theta+\alpha)$

$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$R\cos\alpha = 3 \qquad R\sin\alpha = -2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + (-2)^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 13$

$R = \sqrt{13}, R > 0$

1m for finding R

$\tan\alpha = \frac{-2}{3}$

$\alpha = -0.59$

$\therefore \; 3\sin\theta - 2\cos\theta = \sqrt{13}\sin(\theta - 0.59)$

1m for finding α and writing $3\sin\theta - 2\cos\theta$ in the form $R\sin(\theta+\alpha)$

- many students omitted the final step
- many students found α in degrees

b) $-3\sin\theta + 2\cos\theta = \underbrace{-\sqrt{13}\sin(\theta - 0.59)}_{\text{has max value of } \sqrt{13}}$

1m for writing $-3\sin\theta + 2\cos\theta$ as $-\sqrt{13}\sin(\theta - 0.59)$

$-3\sin\theta + 2\cos\theta - 1 = -\sqrt{13}\sin(\theta - 0.59) - 1$

$\therefore$ max value is $\sqrt{13} - 1$

1m for finding max value

Question 8 :

(a) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ $\qquad x \in [0, 2\pi]$

Solve $\tan^3 x - 3\tan^2 x - 3\tan x + 1 = 0$

$\therefore \quad 1 - 3\tan^2 x = 3\tan x - \tan^3 x$

$\therefore \quad \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = 1$

$\therefore \quad \tan 3x = 1 \qquad x \in [0, 2\pi]$

$\tan 3x = 1 \qquad 3x \in [0, 6\pi]$

$3x = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 4\pi + \frac{\pi}{4}, 5\pi + \frac{\pi}{4}$

$3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

$15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ, 315^\circ$

(b) $x^3 + 5x^2 - 2x + 7 = 0$

$\alpha + \beta + \gamma = -5$

$\alpha\beta + \alpha\gamma + \beta\gamma = -2$

$\alpha\beta\gamma = -7$

$\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\alpha\gamma} + \dfrac{\gamma}{\alpha\beta} =$

$\dfrac{\alpha^2}{\alpha\beta\gamma} + \dfrac{\beta^2}{\alpha\beta\gamma} + \dfrac{\gamma^2}{\alpha\beta\gamma} =$

$\dfrac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma} =$

$\dfrac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)}{\alpha\beta\gamma} =$

$\dfrac{(-5)^2 - 2(-2)}{-7} = \boxed{\dfrac{-29}{7}}$

<table>
<tr><th colspan="3">Year 11 Extension 1: Q9</th></tr>
<tr><th>Suggested Solutions</th><th>Marks Awarded</th><th>Marker's Comments</th></tr>
<tr><td>

(a) Let $t = \theta/2$ with $0 \le \theta \le 2\pi \to 0 \le \frac{\theta}{2} \le \pi$.

Then solve

$$\cos\theta + 3\sin\theta + 1 = \frac{1-t^2}{1+t^2} + \frac{6t}{1+t^2} + \frac{1+t^2}{1+t^2} = 0$$

which gives,

$$2 + 6t = 0$$

or

$$\tan\theta/2 = -\frac{1}{3}$$

So

$$\frac{\theta}{2} = \pi - \tan^{-1}\frac{1}{3}$$

is the only solution from this equation in the given domain; i.e.,

$$\theta = 2\left(\pi - \tan^{-1}\frac{1}{3}\right)$$

Now check for $\theta = \pi$ as possible solution since implicit in using t-sub. is no $\theta = \pi$:

$$\text{LHS} = \cos\pi + 3\sin\pi = -1 = \text{RHS}$$

of given equation. So, $\theta = \pi$ is a solution, too.

Hence the full solution set is:

$$\theta = 2\left(\pi - \tan^{-1}\frac{1}{3}\right),\ \pi$$

</td><td>

1
Correct substitution and $t = -\frac{1}{3}$

1
First correct solution.

1
Second solution (check for π).

</td><td>

Common errors:

- Algebraic missteps leading to $t = 1/3$ or t as a solution to a quadratic equation.
- Incorrect interpretation of arctangent function.
- Not paying attention to domain, giving solutions that are irrelevant.
- Not checking whether $\theta = \pi$ is a solution.
- Mixing degrees and radians, or quoting in degrees completely: the problem is stated in terms of radians, so use radians.
- Please stick to exact values unless otherwise stated. The cohort reported answers in approximate form. Deductions weren't made, obviously, since you need exact form to get to approximate, but unless stated, leave in exact form.

</td></tr>
</table>

(b) Let $h(x) = f(x) - g(x)$. Then $$h(\alpha) = f(\alpha) - g(\alpha) = 0 - 0 = 0$$ since $f(\alpha) = 0 = g(\alpha)$ is given. Hence by the factor theorem, $(x - \alpha)$ is a factor of $h(x)$; i.e., $(x - \alpha)$ is a factor of $f(x) - g(x)$ as required.	1 Showing that the difference at α is zero, or some equivalent move. 1 Using theorem correctly.	Many stated $f(x) = (x-\alpha)Q_1(x)$ and $g(x) = (x-\alpha)Q_2(x)$ as a first step without qualification as to why. The target is known (i.e. that you need $(x-\alpha)$ as a common factor between f and g), so without appeal to the 'factor theorem' or some other proof, there is no justification for this step. It is a statement that you are insisting to be true in order to start, but it's without justification. If you proceed using factorisation as above, you need to make use of the factor theorem to move from knowledge that $f(\alpha) = 0$ to $(x-\alpha)$ being a factor of f. But many people started as: $$f(x) = (x-\alpha)Q(x)$$ $$g(x) = (x-\alpha)P(x)$$ for instance, without any valid reasoning for this initial step. You are required to **show**, which means you're required to justify any claims.

MATHEMATICS Extension 1 : Question 10

Suggested Solutions	Marks	Marker's Comments
i) Since α is a root of the equation:		
$\alpha^2 = 3\alpha - 4$ — (1)	1	
$\therefore \alpha^3 = 3\alpha^2 - 4\alpha \quad (\alpha \neq 0)$ — (2)		
Sub (1) into (2)		
$\alpha^3 = 3(3\alpha - 4) - 4\alpha$		
$= 9\alpha - 12 - 4\alpha$		
$= 5\alpha - 12$	1	
ii) Similarly, $\beta^3 = 5\beta - 12$	1	Note: Students MUST use the method stated in the question. Otherwise 0/3 !
$\therefore \alpha^3 + \beta^3 = (5\alpha - 12) + (5\beta - 12)$	1	
$= 5(\alpha + \beta) - 24$		
$= 5 \times 3 - 24$ (Sum of roots)		
$= -9$	1	

Suggested Solutions	Marks	Marker's Comments
Alternate Method 2:		
$\alpha+\beta=3$ $\alpha\beta=4$	1	
Since α is a root to $x^2-3x+4=0$		
then $\alpha^2-3\alpha+4=0$ $\therefore \alpha^2=3\alpha-4$ $=3\alpha-4+3\beta-3\beta$ $=3(\alpha+\beta)-3\beta-4$ $=9-3\beta-4$ $=5-3\beta$		
$\alpha^3=5\alpha-3\alpha\beta \quad (\alpha\neq 0)$ $=5\alpha-3(4)$ $=5\alpha-12$	1	

MATHEMATICS EXT 1: Question 11

Suggested Solutions	Marks	Marker's Comments
(a) $\sin\left(\theta - \frac{\pi}{3}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \frac{1}{2} \quad x \in [0, 2\pi]$		Should just state something like "by sum-to-product". Always aim to justify your non-trivial steps.
$2\sin\theta\cos\frac{\pi}{3} = \frac{1}{2}$ using $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$	1	
$2\sin\theta \times \frac{1}{2} = \frac{1}{2}$		
$\sin\theta = \frac{1}{2}$		
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$	1	
Note: $\sin\theta = \frac{1}{2}$, $\theta = \sin^{-1}\left(\frac{1}{2}\right)$, $\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ — This is WRONG! If you did this, you don't understand that $\sin^{-1}x$ is one-to-one.		Similar errors in part (bi)
(bi) We can check $x = \sin\frac{\pi}{10}$ and $x = \sin\frac{13\pi}{10}$ are solutions by showing they satisfy the equation.		
• sub $x = \sin\frac{\pi}{10}$ into LHS		
$LHS = \left(16\sin^5\frac{\pi}{10} - 20\sin^3\frac{\pi}{10} + 5\sin\frac{\pi}{10}\right) - 1$		
$= \sin\left(5 \times \frac{\pi}{10}\right) - 1$, from given fact		
$= \sin\left(\frac{\pi}{2}\right) - 1$		
$= 1 - 1$		
$= 0$		
$= RHS \Rightarrow x = \sin\frac{\pi}{10}$ is a solution	1	

MATHEMATICS EXT 1: Question		
Suggested Solutions	**Marks**	**Marker's Comments**
• sub $x = \sin\frac{13\pi}{10}$ into LHS		
$LHS = \left(16\sin^5\frac{13\pi}{10} - 20\sin^3\frac{13\pi}{10} + 5\sin\frac{13\pi}{10}\right) - 1$		
$= \sin\left(5\times\frac{13\pi}{10}\right) - 1$, from given fact		
$= \sin\left(\frac{13\pi}{2}\right) - 1$		
$= \sin\left(6\pi + \frac{\pi}{2}\right) - 1$		
$= \sin\left(\frac{\pi}{2}\right) - 1$		
$= 1 - 1$		
$= 0$		
$= RHS \Rightarrow x = \sin\frac{13\pi}{10}$ is a solution	1	
ALTERNATIVELY:		
We have $16x^5 - 20x^3 + 5x - 1 = 0$		MUST sub in $x=\sin\theta$. Can't just arrive at $\sin 5\theta = 1$ out of nowhere.
Let $x = \sin\theta$		
Then, $(16\sin^5\theta - 20\sin^3\theta + 5\sin\theta) - 1 = 0$		
$\sin 5\theta - 1 = 0$ using given fact	1	
$\sin 5\theta = 1$		
$5\theta = \frac{\pi}{2} + 2k\pi$, for $k \in \mathbb{Z}$		
$\theta = \frac{\pi}{10} + \frac{2k\pi}{5}$, for $k \in \mathbb{Z}$		You should specify $k \in \mathbb{Z}$. Not penalised this time, but be rigorous!!
When $k = 0$: $\theta = \frac{\pi}{10}$		
When $k = 3$: $\theta = \frac{13\pi}{10}$		
$\therefore x = \sin\frac{\pi}{10}$ & $x = \sin\frac{13\pi}{10}$ are solutions.	1	

MATHEMATICS EXT 1: Question

Suggested Solutions	Marks	Marker's Comments
(bii) $(x-1)(4x^2+ax-1)^2 = 16x^5-20x^3+5x-1$		Many went for long division which was long & messy.
$(x-1)(4x^2+ax-1)(4x^2+ax-1) = 16x^5-20x^3+5x-1$		Comparing coeffs / "inspection" is more efficient
Compare coeff. of x: $1+a+a=5$		
$2a=4$		
$a=2$	1	
Since $x=\sin\frac{\pi}{10}, \sin\frac{13\pi}{10}$ are solutions of $16x^5-20x^3+5x-1=0$, and neither are equal to 1, then they must be solutions of $(4x^2+2x-1)^2=0$.		must have that $\sin\frac{\pi}{10}\neq 1$, $\sin\frac{13\pi}{10}\neq 1$.
Also, since $\sin\frac{\pi}{10}\neq\sin\frac{13\pi}{10}$, they are both double roots of $(4x^2+2x-1)^2=0$.		
$\therefore$ the quadratic $4x^2+2x-1=0$ has roots $x=\sin\frac{\pi}{10}, \sin\frac{13\pi}{10}$	1	
$\sum\alpha = \frac{-b}{a}$		
$\sin\frac{\pi}{10}+\sin\frac{13\pi}{10} = -\frac{1}{2}$	1	
$\sin\frac{\pi}{10}+\sin\left(\pi+\frac{3\pi}{10}\right) = -\frac{1}{2}$		
$\sin\frac{\pi}{10}-\sin\frac{3\pi}{10} = -\frac{1}{2}$	1	Must justify the $\sin\frac{3\pi}{10}$ part. Can't go from $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}=-\frac{1}{2}$ to $\sin\frac{\pi}{10}-\sin\frac{3\pi}{10}=-\frac{1}{2}$ without any explanation.