Student Name: ____________________

Maths class: ____________________

James Ruse Agricultural High School

2021 **YEAR 12 Task 2**

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 6 - 13, show relevant mathematical reasoning and/ or calculations

Total marks: 79

Section I – 5 marks (pages 2 - 3)

- Attempt Questions 1–5
- Allow about 7 minutes for this section

Section II – 74 marks (pages 4 - 9)

- Attempt Questions 6 - 13
- Allow about 1 hour and 53 minutes for this section

Section I

Write your answers on the multiple choice answer sheet provided.

1. Which of the following is equivalent to $\int_0^{0.5} \frac{e^{2x}}{\sqrt{1+e^{2x}}} dx$ when the substitution $u = e^{2x}$ is made?

A. $\int_1^e \frac{u}{\sqrt{1+u^2}} du$

B. $\frac{1}{2}\int_1^e \frac{1}{\sqrt{1+u}} du$

C. $\int_0^{0.5} \frac{u}{\sqrt{1+u^2}} du$

D. $\frac{1}{2}\int_0^{0.5} \frac{1}{\sqrt{1+u}} du$

2. Ten different letters of the alphabet are given. Words with six letters are formed from these given letters, where letters may possibly be repeated.

Find the number of words which have at least one letter repeated?

A. ${}^{10}C_6$

B. 10^6

C. $10^6 - {}^{10}P_6$

D. ${}^{10}P_6$

3. The value of

$$\int_{-1}^{1} \cos^{-1} x \, dx$$

is:

A. 0

B. $\frac{\pi}{2}$

C. π

D. 2π

4. A solid is formed by rotating the graph $y = \sqrt{x}$ on the interval $[2, 4]$ about the x-axis.

For what value of k does the line $x = k$ divide the solid into two solids with equal volumes?

A. 3.033

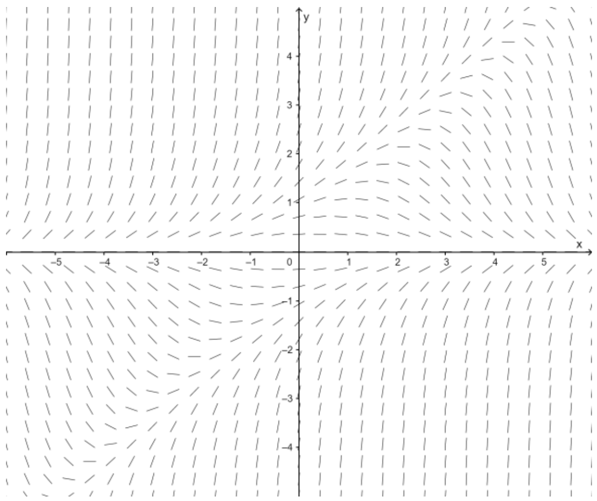
B. 3.051

C. 3.108

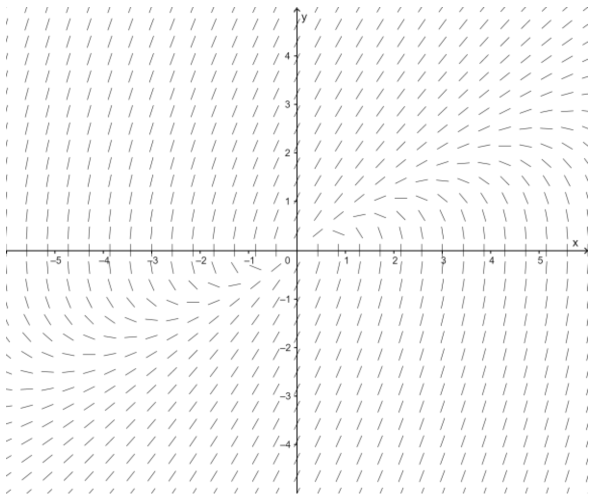
D. 3.162

5. Which of the following best represents the direction field for the differential equation $\frac{dy}{dx} = \frac{x-2y}{x}$?

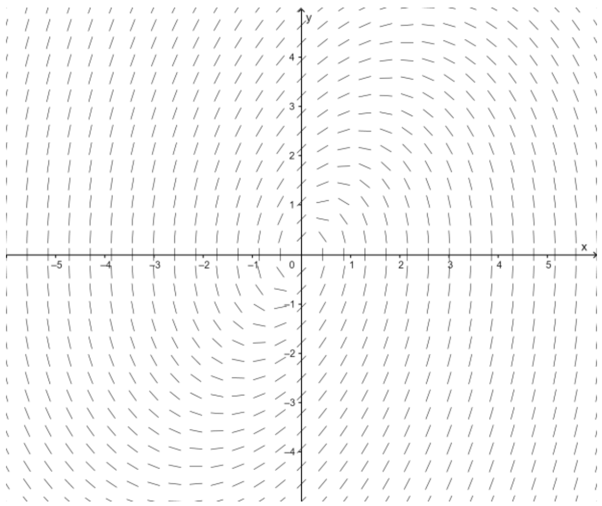
A.



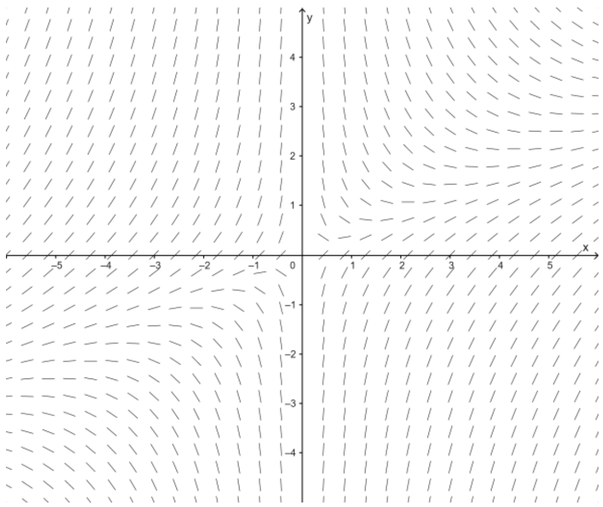
B.



C.



D.



Section II begins on the next page

Section II

74 marks

Attempt Questions 6 - 13

Allow about 1 hour and 53 minutes for this section

Start each question on a new page. Extra paper is available.

In Questions 6 - 13, your responses should include relevant mathematical reasoning and/ or calculations.

Question 6 (10 marks) **Start a new page.**

(a) Find $\int \sin^2 3x \, dx$. **2**

(b) If $y = \sin^{-1} x$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ by first finding $\frac{dx}{dy}$ in terms of y. **3**

(c) Find the exact value of the volume of the solid generated when the region bounded by the graphs of $y = \sqrt{x},\ y = 2 - x$ and $x = 0$ is rotated about the x-axis. **3**

(d) An employee asked 10 people in an ice cream shop to wait in line, 1.5m apart. Determine the number of different arrangements possible if two of the people, Joe and Jonah, refused to stand next to each other in the line. **2**

Question 7 (10 marks) **Start a new page.**

(a) Use the substitution $x = \tan^{-1} u$ to evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x} dx$. **3**

(b) Suppose the birthrate for a certain population is given by $b(t) = 2e^{0.04t}$ million people per year and the death rate for the same population is $d(t) = 2e^{0.02t}$ million people per year.

(i) Given $b(t) \geq d(t)$ for $t \geq 0$ explain why the area between the curves represents the increase in population. **1**

(ii) Calculate the increase in population for $0 \leq t \leq 10$ **1**

(c) Let $\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$, express y in term of x, where $y\left(\frac{\pi}{2}\right) = 0$. **2**

(d) In how many ways can the diagonals of a decagon dissect the decagon into two pieces, neither of which is a triangle, by drawing a straight line joining two vertices? **3**

Question 8 (9 marks) **Start a new page.**

(a) (i) Find the equation of the tangent to the curve $y = \tan^{-1} x$ at $x = \frac{\pi}{n}$, $n \in \mathbb{R}$. **2**
Express your answer in gradient intercept form.

(ii) Hence or otherwise, write down the y-intercept of the tangent as $n \to 0^+$. **1**

(b) Sketch the region and axis of revolution that produces the solid with volume given by **2**

$$V = \int_0^2 \pi(\sin^{-1}(x-1))^2 dx$$

(c) Find the curve that satisfies the differential equation $\frac{dy}{dx} = \frac{(2y-1)}{\sqrt{4-9x^2}}$ and passes through the point $\left(0, -\frac{1}{2}\right)$. **4**

Express y in terms of x.

Question 9 (9 marks) **Start a new page.**

(a) The region under $y = \frac{1}{x^2+1}$ above the x–axis and between the lines $x = 0$ and $x = 1$ is revolved about the x-axis.

(i) Show that the volume of the solid formed is given by $V = \pi \int_0^1 \frac{dx}{(x^2+1)^2}$. **1**

(ii) By using the substitution $x = \tan\theta$, find the exact volume of the solid. **3**

(b) In psychology, the learning curve is a graphical representation of how long it takes a person to acquire new skills or knowledge.

Let $L(t)$ represent the performance of a person learning a skill and t be the training time.

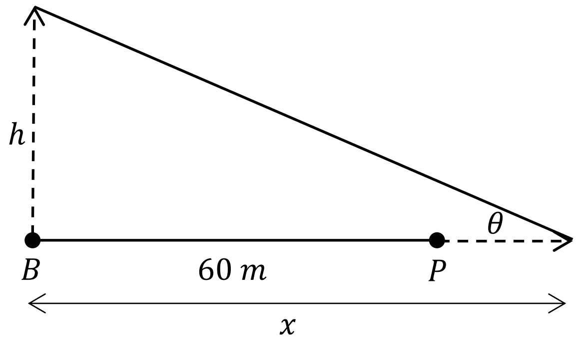
The differential equation $\frac{dL}{dt} = k(M-L)$, where k is a positive constant and M is the maximum level of performance of which the learner is capable, applies.

(i) If the initial condition is $L(0) = N$, derive $L(t) = M + (N-M)e^{-kt}$ **3**

(ii) If $N = 0.2\,M$, how long will it take a learner to reach three quarters of the maximum level of performance M? **2**

Question 10 (9 marks) **Start a new page.**

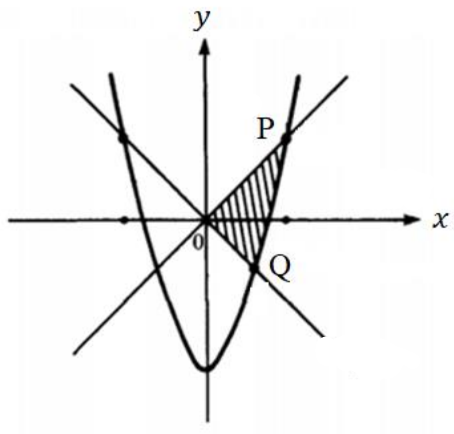
(a) A helicopter leaves a base B, rising straight up at a constant speed of $10\ m$/s. At the same time the helicopter leaves, an observer starts from a point P, $60\ m$ away from the base, and moves in a straight line away from the base at a constant speed of $2\ m/s$. After t seconds, the distance between the observer and the base is x metres, and the angle of elevation of the helicopter, h metres above the base, from the observer, is θ.



(i) Write expressions for x and h in terms of t respectively. **2**

(ii) Find the rate at which the angle of elevation is changing when the observer is $70\ m$ away from the base. Give your answer correct to 2 significant figures in radians per second. **3**

(b) Consider the region bounded by the curve $y = x^2 - 6$ and the lines $y = x$ and $y = -x$ for $x \geq 0$ as shown in the diagram below. **2**



Given the x-coordinates of P and Q are 3 and 2 respectively find the exact area of the shaded region.

(c) Six dice are rolled. What is the probability of getting three pairs (that is, three different numbers that each appear twice)? **2**

Question 11 (9 marks) **Start a new page.**

(a) Determine the exact area of the region bounded by $y = e^{1+2x},\ y = e^{1-x},\ x = -2$ and $x = 1$. **3**

(b) A large tank holds 600 L of water in which 15 kg of salt has been dissolved.
Initially pure water starts to flow into the tank at a rate of 10 L per minute.
The mixture is stirred continuously and flows out of the tank at a rate of 10 L per minute.

Let m be the mass of salt at time t.

(i) Show that the differential equation to model the rate of change of the mass of the salt is given by $\frac{dm}{dt} = -\frac{m}{60}$. **1**

(ii) Solve the differential equation in (i) to express m in terms of t. **2**

(iii) Half an hour later, pure water is replaced with seawater of 1 kg/L concentration, flowing into the tank at 10 L per minute. **1**

Construct another differential equation to model the rate of change of the mass of the salt. Assume the same outflow rate as in (i).

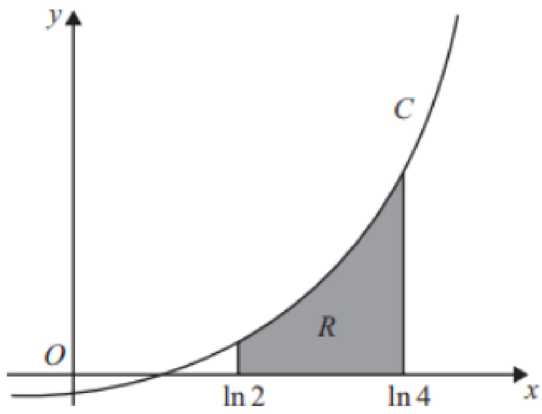
(iv) Hence express t in terms of m. **2**

Question 12 (9 marks) **Start a new page.**

(a) The curve C has parametric equations $x = \ln t\,,\;\; y = t^2 - 2,\;\; t > 0$.

(i) Find a cartesian equation of C. **1**

(ii) The finite area R shown in the diagram is bounded by C, the x-axis and the line $x = \ln 2$ and the line $x = \ln 4$. **3**



Find the exact volume of the solid of revolution formed when the area R is rotated about the x-axis.

(b) The growth of the population of penguins on an isolated island is modelled by the differential equation

$$\frac{dN}{dt} = \frac{0.02N(8-N)}{8}$$

where N is the population of penguins at time t.

(i) If initially there are 7 penguins, solve the differential equation, to find N as a function of t. **3**

You may assume: $\frac{k}{N(k-N)} = \frac{1}{N} + \frac{1}{k-N}$, where $k \in \mathbb{R}$

(ii) Hence determine the limiting value of the penguin population on this island? **2**

Justify your answer.

Question 13 (9 marks) **Start a new page.**

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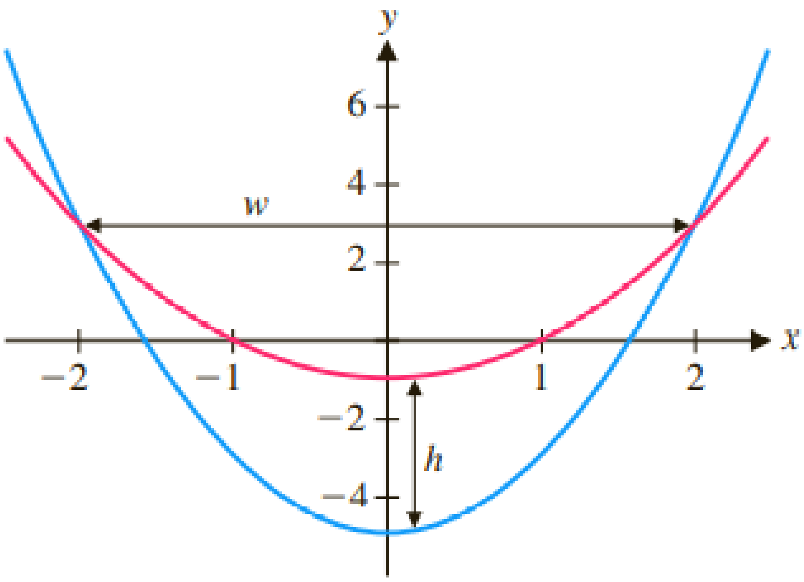
(a) Solve $\frac{dy}{dx} = \frac{1}{2}\cos 2x \sin 3x$ **3**

(b) Consider two parabolas, each of which has its vertex at $x = 0$, but with different y-intercepts and different concavities. **3**

Let h be the difference in y-coordinates of the vertices, and let w be the difference in the x-coordinates of the intersection points.



Show that the area between the curves is $\frac{2}{3}hw$.

(c) A woman has a certain number of friends; she wants to invite three of them for dinner every day of the year. **3**

What is the least number of friends she must have if she does not want to invite the same three friends twice?

Ext 1 2021 Yr 12 Task 2

MATHEMATICS Extension 1 : Question MC

Suggested Solutions	Marks	Marker's Comments

1. B

$u = e^{2x}$

$\frac{du}{dx} = 2e^{2x}$

$\frac{1}{2}du = e^{2x}dx$

when $x = 0.5$, $u = e^{2\times 0.5} = e$

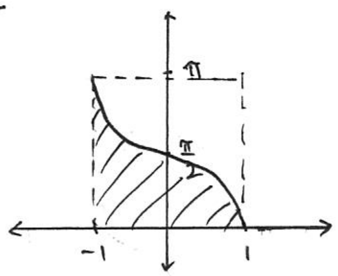
when $x = 1$, $u = e^{2\times 0}$, $u = 1$

$\therefore \int_0^{0.5} \frac{e^{2x}}{\sqrt{1+e^{2x}}}dx$

$= \int_1^e \frac{\frac{1}{2}du}{\sqrt{1+u}}$

2. C

3. C



Area is half of the rectangle

$= \frac{1}{2} \times (2\times\pi)$

$= \pi$

4. D

$\pi\int_2^k (\sqrt{x})^2 dx = \pi\int_k^4 (\sqrt{x})^2 dx$

$\int_2^k x\,dx = \int_k^4 x\,dx$

$\left[\frac{x^2}{2}\right]_2^k = \left[\frac{x^2}{2}\right]_k^4$

$\frac{k^2}{2} - \frac{2^2}{2} = \frac{4^2}{2} - \frac{k^2}{2}$

$k^2 = 8 + 2$

$k = \sqrt{10}$

$k = 3.1622\ldots$

5. D

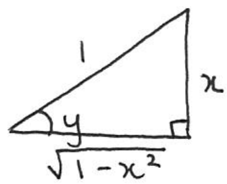
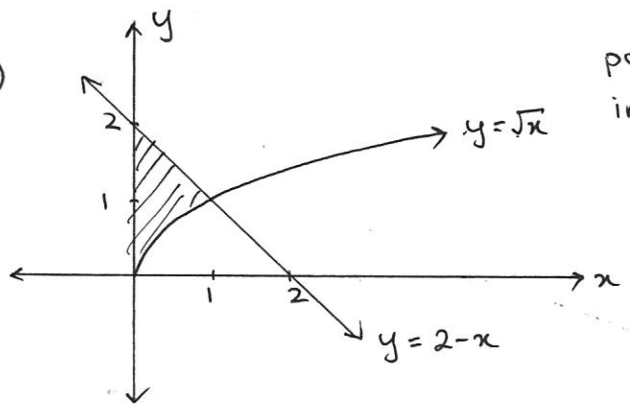
when $y = 0$

$\frac{dy}{dx} = \frac{x - 2(0)}{x}$

$= 1$

$\therefore$ gradient along the x-axis is 1.

MATHEMATICS Extension 1 : Question 6

Suggested Solutions	Marks	Marker's Comments
a) $\int \sin^2 3x\,dx = \int \frac{1}{2}(1-\cos 6x)\,dx$	1	$\sin^2 nx = \frac{1}{2}(1-\cos 2nx)$ on the reference sheet (students still got it wrong)
$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + C$	1	- must have +c !
b) $y = \sin^{-1} x$		
$x = \sin y$		
$\frac{dx}{dy} = \cos y$	1	
$\frac{dy}{dx} = \frac{1}{\cos y}$	1	for the reciprocal
	1	for the triangle
$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$		
OR. $\frac{dx}{dy} = \cos y$	1	
$= \sqrt{\cos^2 y}$		
$= \sqrt{1-\sin^2 y}$	1	for $\cos^2 y = 1 - \sin^2 y$
$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$	1	for the reciprocal
$= \frac{1}{\sqrt{1-x^2}}$		
c)  point of intersection (1,1)		A lot of students used the incorrect region! Answer for incorrect region is $\frac{5\pi}{6}$

MATHEMATICS Extension 1 : Question...6...

Suggested Solutions	Marks	Marker's Comments
$V = \pi \int_0^1 \left((2-x)^2 - (\sqrt{x})^2\right) dx$	1	correct equation with boundaries.
$= \pi \int_0^1 (4 - 4x + x^2 - x)\, dx$		
$= \pi \int_0^1 (x^2 - 5x + 4)\, dx$		
$= \pi \left[\frac{x^3}{3} - \frac{5}{2}x^2 + 4x \right]_0^1$	1	
$= \pi \left[\frac{1^3}{3} - \frac{5}{2}(1) + 4(1) - 0 \right]$		
$= \frac{11\pi}{6} u^3$	1	
d) total arrangements $= 10!$	1	
Joe and Jonah next to each other $= 2! \times 9!$		
$\therefore$ Joe and Jonah not next to each other $= 10! - 2! \times 9!$		
$= 2903040$	1	
<u>OR</u> _ _ _ _ _ _ _ _		
8 not including Joe and Jonah $= 8!$	1	
gaps for Joe and Jonah $= 9 \times 8$		
$\therefore$ total $= 8! \times 9 \times 8$		
$= 2903040$	1	

Question 7 marking scheme

Tuesday, 6 April 2021 1:48 pm

MATHEMATICS Extension 1: Question 7

Suggested Solutions	Marks Awarded	Marker's Comments
a) $u=\tan x$ $\frac{du}{dx}=\sec^2 x$ $=1+\tan^2 x$ $=1+u^2$		
when $x=\pi/4$, $u=1$ when $x=\pi/3$, $u=\sqrt{3}$	1	$\leftarrow$ bounds in terms of u
$\int_{\pi/4}^{\pi/3} \frac{\sec^2 x}{\tan x}\,dx = \int_1^{\sqrt{3}} \frac{1+u^2}{u}\cdot\frac{du}{1+u^2}$	1	$\leftarrow$ integral in terms of u
$=\int_1^{\sqrt{3}} \frac{1}{u}\,du$ $=\left[\ln\lvert u\rvert\right]_1^{\sqrt{3}}$		
$=\ln\sqrt{3}-\ln 1$ $=\ln\sqrt{3}$	1	$\leftarrow$ showing substitution of $u=\sqrt{3}$ and $u=1$.
b) (i) Area under $b(t)$ represents no. of births Area under $d(t)$ represents no. of deaths $b(t)-d(t)\geq 0$ $\therefore$ Area between curves represents an increase in population.	1	Must explain area under curve is no. of births / deaths
(ii) Increase in population $=\int_0^{10} 2e^{0.04t}-2e^{0.02t}\,dt$ $=2\left[\frac{1}{0.04}e^{0.04t}-\frac{1}{0.02}e^{0.02t}\right]_0^{10}$ $=2\left[25e^{0.04t}-50e^{0.02t}\right]_0^{10}$ $=2\left[25e^{0.4}-50e^{0.2}-25+50\right]$		
$=2.451$ million	1	Millions must be stated.

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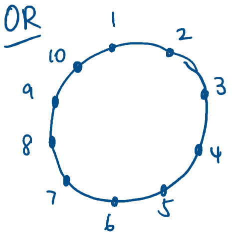
MATHEMATICS Extension 1 : Question 7

<table>
<tr><th>Suggested Solutions</th><th>Marks Awarded</th><th>Marker's Comments</th></tr>
<tr><td>c) $\frac{dy}{dx} = \frac{\cos x}{1+\sin x}$
$y = \int \frac{\cos x}{1+\sin x}\,dx$
$= \ln\lvert 1+\sin x\rvert + C$</td><td>1 ⟵</td><td>must have +C</td></tr>
<tr><td>when $x = \pi/2$, $y = 0$
$\Rightarrow C = -\ln 2$
$\therefore y = \ln\lvert 1+\sin x\rvert - \ln 2$
$= \ln\left\lvert \frac{1+\sin x}{2}\right\rvert$</td><td>1 ⟵</td><td>C correctly evaluated.</td></tr>
<tr><td>d) Choose any two vertices in ${}^{10}C_2$ ways</td><td>1</td><td></td></tr>
<tr><td>No. of pairs of vertices next to each other $= 10$.</td><td>1</td><td></td></tr>
<tr><td>No. of pairs of vertices one apart $= 10$</td><td>1</td><td></td></tr>
<tr><td>$\therefore$ No. of ways $= {}^{10}C_2 - 10 - 10$
$= 25$</td><td></td><td></td></tr>
<tr><td>OR
$\frac{10\times(10-5)}{2}$</td><td>1 ⟵</td><td>Choose from 10 vertices</td></tr>
<tr><td>= no. of vertices × no. of vertices can connect to ÷ 2</td><td>1 ⟵</td><td>no. of vertices can connect to = 5</td></tr>
<tr><td>= 25</td><td>1 ⟵</td><td>divide by 2</td></tr>
<tr><td>OR


<table>
<tr><th>vertex</th><th>no. of vertices can connect to</th></tr>
<tr><td>1</td><td>5</td></tr>
<tr><td>2</td><td>5</td></tr>
<tr><td>3</td><td>5</td></tr>
<tr><td>4</td><td>4</td></tr>
<tr><td>5</td><td>3</td></tr>
<tr><td>6</td><td>2</td></tr>
<tr><td>7</td><td>1</td></tr>
<tr><td>Total</td><td>25</td></tr>
</table></td><td>3 ⟵

2 ⟵

1 ⟵</td><td>correctly adding

one incorrect no. of vertices

two incorrect no. of vertices.</td></tr>
</table>

(Q8) ai, $y = \tan^{-1} x$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$x = \frac{\pi}{n} \rightarrow \frac{dy}{dx} = \frac{1}{1+(\pi/n)^2}$$ .... (1)

$$= \frac{n^2}{n^2+\pi^2}$$

- ANY correct expression for the gradient.

Eqn of tangent at $\left(\frac{\pi}{n}, \tan^{-1}\left(\frac{\pi}{n}\right)\right)$

$$y - \tan^{-1}\left(\frac{\pi}{n}\right) = \frac{n^2}{n^2+\pi^2}\left(x - \frac{\pi}{n}\right)$$ .... (1)

$$y = \frac{n^2}{n^2+\pi^2} x + \left(\tan^{-1}\left(\frac{\pi}{n}\right) - \frac{n\pi}{n^2+\pi^2}\right)$$

- ANY correct form for a line
- Ignoring $y = mx + b$

ii, y-intercept of tangent is given by:

$$b = \tan^{-1}\left(\frac{\pi}{n}\right) - \frac{n\pi}{n^2+\pi^2}$$

when $n \rightarrow 0^+$, $b \rightarrow \frac{\pi}{2} - 0$

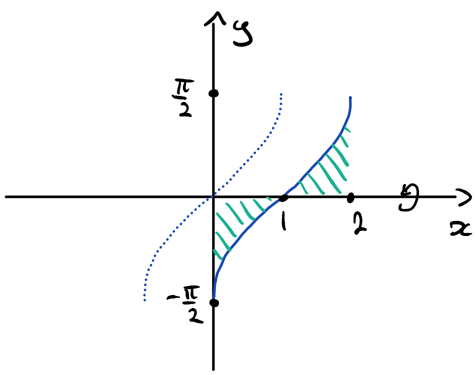
$$= \pi/2$$ .... (1)

- You could have arrived at the same result graphically.
- If you made a mistake in part (i) but correctly evaluated a limit, you received the mark.

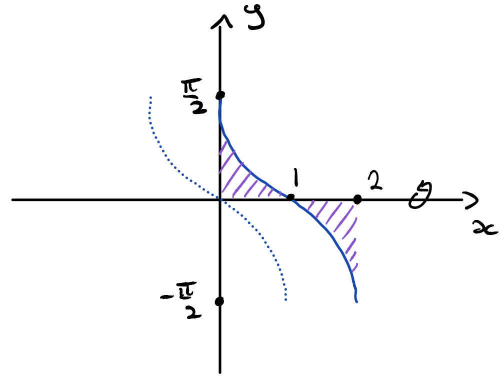
b) $V = \int_0^2 \pi \left(\sin^{-1}(x-1)\right)^2 dx$

$\therefore [f(x)]^2 = \left(\sin^{-1}(x-1)\right)^2$

$\therefore f(x) = \pm \sin^{-1}(x-1)$



OR



correct sketch
correct region shaded
correct axis of revolution

② all three
① two of the three.

- If you wrote $y = \sin^{-1}(x-1)$ but drew $y = -\sin^{-1}(x-1)$, no mark is awarded.
- Any indication of the correct axis of revolution was awarded a mark, but you should explicitly state it.

c, $\dfrac{dy}{dx} = \dfrac{(2y-1)}{\sqrt{4-9x^2}}$

Separating variables

$\frac{1}{2}\int \dfrac{2\,dy}{2y-1} = \frac{1}{3}\int \dfrac{3\,dx}{\sqrt{4-9x^2}}$ 1

$\frac{1}{2}\ln|2y-1| = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C$ 1

Correctly integrating

when $x=0,\ y=-\frac{1}{2} \rightarrow \frac{1}{2}\ln|-2| = 0 + C$

$C = \frac{1}{2}\ln 2$ 1

Correctly evaluating integration constant

$\therefore \frac{1}{2}\ln|2y-1| = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + \frac{1}{2}\ln 2$

$\frac{1}{2}\ln\left|\frac{2y-1}{2}\right| = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)$

$\ln\left|\frac{2y-1}{2}\right| = \frac{2}{3}\sin^{-1}\left(\frac{3x}{2}\right)$

$\left|\frac{2y-1}{2}\right| = e^{2/3\sin^{-1}\left(\frac{3x}{2}\right)}$

$\frac{2y-1}{2} = \pm e^{2/3\sin^{-1}\left(\frac{3x}{2}\right)}$

$y = \frac{1}{2} \pm e^{2/3\sin^{-1}\left(\frac{3x}{2}\right)}$

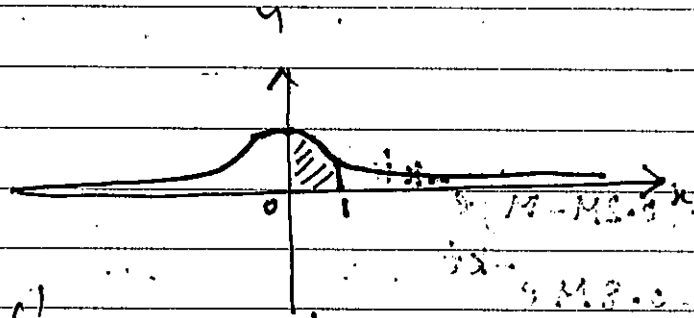
but $y(0) = -\frac{1}{2}$

$\therefore y = \frac{1}{2} - e^{\frac{2}{3}\sin^{-1}\left(\frac{3x}{2}\right)}$ 1

correct justification of plus/minus.

Question 9:

(a)



(i) $V = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 \frac{dx}{(x^2+1)^2}$

(ii) let $x = \tan\theta \quad \therefore 1 + x^2 = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$V = \pi \int_0^{\pi/4} \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = \pi \int_0^{\pi/4} \cos^2\theta \, d\theta$

$= \pi \int_0^{\pi/4} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta$

$= \pi \left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{\pi/4}$

$= \pi \left[\frac{1}{4} + \frac{\pi}{8} - 0\right]$

$= \pi \left[\frac{1}{4} + \frac{\pi}{8}\right] U^3$

(b) (i) $\frac{dL}{dt} = k(M - L)$

$\int \frac{dL}{M-L} = \int k \, dt$

$-\ln|M-L| = kt + c$

$\ln|M-L| = -kt - c$

$M - L = e^{-kt} \cdot e^{-c} = Ae^{-kt}$

$L = M - Ae^{-kt}$

$L(0) = N = M - A \therefore A = M - N$

$L = M + (N - M)e^{-kt}$

(ii) $N = 0.2M$

$L = \frac{3}{4}M$

$\frac{3}{4}M = M + (0.2M - M)e^{-kt}$

$-\frac{1}{4}M = -0.8Me^{-kt}$ ✓

$e^{-kt} = \frac{1}{4 \times 0.8}$

$-kt = -\ln(4 \times 0.8)$

$t = \frac{1}{k}\ln(3.2)$ ✓

Mathematics Extension 1: Question 10		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
(a) (i) $x = 60 + 2t, \quad h = 10t$	**2**	Question does not imply derivation needed via calculus. The rates here are constant.
(ii) Require $\frac{d\theta}{dt}$ when $x = 70$ metres. Now, $$\tan\theta = \frac{h}{x} = \frac{10t}{60+2t}$$ So $$\theta = \tan^{-1}\left(\frac{10t}{60+2t}\right)$$	**1**	Many students took a derivative with respect to x or to h without recognizing that if done so, x would be a function of h or h would be a function of x. Consequently, the derivative was incorrect.
Hence $$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{10t}{60+2t}\right)^2}\cdot\frac{d}{dt}\left(\frac{10t}{60+2t}\right)$$ $$= \frac{1}{1+\left(\frac{10t}{60+2t}\right)^2}\cdot\frac{10(60+2t)-10t(2)}{(60+2t)^2}$$ $$= \frac{600}{(60+2t)^2+100t^2}$$	**1**	E.g. $$\frac{d(hx^{-1})}{dx} = h\frac{dx^{-1}}{dx} + x^{-1}\frac{dh}{dx}$$ is correct, but students took h to be independent of x, thereby giving $$\frac{d(hx^{-1})}{dx} = h\frac{dx^{-1}}{dx}$$
When $x = 70$, $t = 5$, so $$\frac{d\theta}{dt} = \frac{3}{37} \approx 0.081$$ radians per second.	**1**	which is false.
(b) Area obtained from considering area between the two lines from origin to x-coordinate of Q (area of triangle), then between the line OP and quadratic from x-coordinate of Q to x-coordinate of P.		Approximately 47% of the candidature failed to get full marks in this question.
Since Q lies on $y = -x$, $Q = (2, -2)$ and $y = x$ is the reflection of OQ about the x-axis. Hence the area of the triangle OQQ' is $$\frac{1}{2}(\text{base})\times(\text{height}) = \frac{1}{2}(4)(2) = 4$$ square units.	**1**	Students did not recognise the most efficient means of partitioning, or did not partition correctly, or did not understand that if $f(x) \geq g(x)$ over and

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Could also find via integration:

$$\int_0^2 x-(-x)\ dx = \int_0^2 2x\ dx = x^2|_0^2 = 4$$

Next, area between line OP and parabola:

$$\int_2^3 x-(x^2-6)\ dx = x^2 - \frac{x^3}{3} + 6x\ |_2^3 = \frac{13}{6}$$

Hence the total area is

$$4+\frac{13}{6}=\frac{37}{6}$$

square units.

(c) Since all outcomes are equally likely, we need only count the favourable outcomes and divide by the total possible outcomes.

An example of a favourable outcome is

<u>1</u> <u>1</u> <u>2</u> <u>2</u> <u>3</u> <u>3</u>

where position 1 is the outcome on die 1, position 2 is the outcome on die 2, etc.

Note that this instance has us

(1) selecting three distinct numbers from six to make our pairs,
(2) permuting the result amongst the six possible places.

For (1), we need to choose three numbers from a set of six: $\binom{6}{3}$.

And for (2), for each of the selections made in (1), we need to permute the pairs as we would a six-letter word containing three pairs of the same letter:

$$\frac{6!}{(2!)^3}$$

Hence the total number of favourable outcomes is

$$\binom{6}{3}\cdot\frac{6!}{(2!)^3}=1800$$

The total number of possible outcomes is 6^6 since for each die, there are six selections. Hence the probability required is

$$\frac{1800}{6^6}=\frac{25}{648}$$

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1

1

1

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interval, then the area between is always the integral of $f(x)-g(x)$ (i.e. upper minus lower, no matter what).

One mark was awarded for relevant, constructive attempt.

Not many survived this question. It was obvious that students did not have a structure or a plan of attack.

One mark was awarded for *<u>relevant</u>*, constructive attempt **that included a written indication of thought process (i.e. sentences, not just numbers).**

No marks awarded if the result was incorrect and there was no communication as to thought process.

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a) Shaded Area

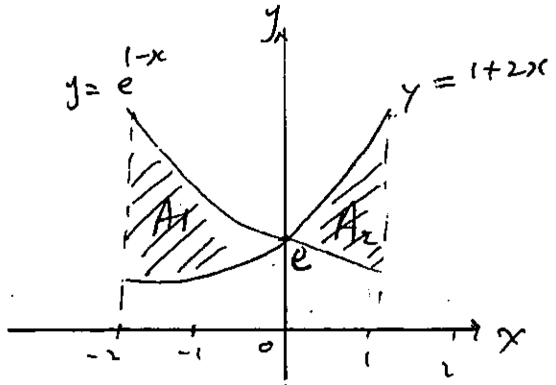
$$= \int_{-2}^{0} e^{1-x} - e^{1+2x}\,dx + \int_{0}^{1} e^{1+2x} - e^{1-x}\,dx$$

$$= \left(-e^{1-x} - \frac{e^{1+2x}}{2}\right)\Big]_{-2}^{0} + \left(\frac{e^{1+2x}}{2} + e^{1-x}\right)\Big]_{0}^{1}$$

$$= -\left(e + \frac{e}{2}\right) + \left(e^3 + \frac{e^{-3}}{2}\right) + \left(\frac{e^3}{2} + e^0 - \frac{e}{2} - e^1\right)$$

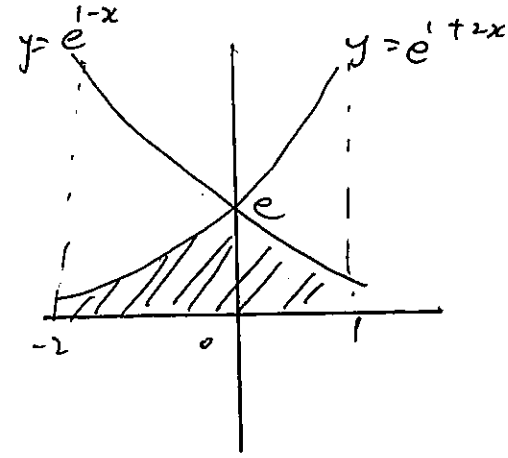
$$= \underbrace{\left(-e - \frac{e}{2} + e^3 + \frac{1}{2e^3}\right)}_{(A_1)} + \underbrace{\left(\frac{e^3}{2} + 1 - \frac{e}{2} - e\right)}_{(A_2)}$$

$$= \frac{3}{2}e^3 - 3e + 1 + \frac{1}{2e^3} \text{ unit}^2 \ \#$$



Students with wrong graphs or can't separate limits get 0 m

many students mistook the Area as →



$$\text{Area} = \int_{-2}^{0} e^{1+2x}\,dx + \int_{0}^{1} e^{1-x}\,dx$$

$$= \frac{1}{2}e^{1+2x}\Big]_{-2}^{0} + \left(-e^{1-x}\right)\Big]_{0}^{1}$$

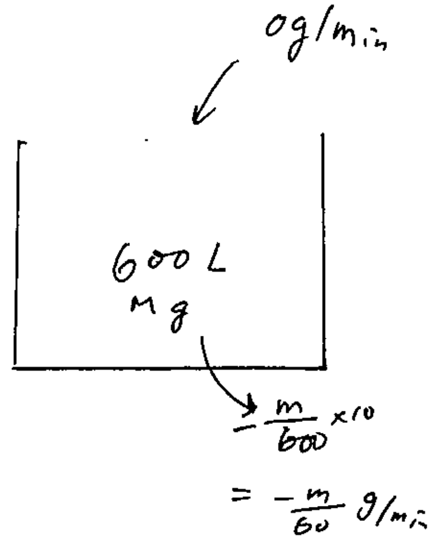
$$= \frac{1}{2}\left(e - e^{-3}\right) - \left(e^0 - e\right)$$

$$= \frac{e}{2} - \frac{1}{2e^3} - 1 + e$$

$$= \frac{3e}{2} - \frac{1}{2e^3} - 1 \text{ unit}^2$$

max 2m

b) i) $\frac{dm}{dt} = 0 - \frac{m}{60} = \frac{-m}{60}$



ii) $t=0, \quad m=15$

$$\frac{dm}{m} = -\frac{dt}{60}$$

$$\ln|m| = \frac{-t}{60} + C$$

$m>0, \quad \ln m = \frac{-t}{60} + C$

$t=0,\ m=15$

$\therefore$

$\ln 15 = c$

$\therefore\ \ln m = -\frac{t}{60} + \ln 15$

$\frac{m}{15} = e^{-\frac{t}{60}}$ $\qquad$ $\underline{m = 15e^{-t/60}}$

iii) $\frac{dm}{dt} = 10 - \frac{m}{600}\times 10 = 10 - \frac{m}{60}$ or $\frac{600-m}{60}$

iv) $t=30$ $\quad m = 15e^{-30/60} = 15e^{-\frac{1}{2}}$

$\int \frac{dm}{600-m} = \int \frac{dt}{60}$

$-\ln|600-m| = \frac{t}{60} + C$

$t=30 \quad m = 15e^{-\frac{1}{2}}$

$-\ln|600-15e^{-\frac{1}{2}}| = \frac{30}{60} + c$

$-\frac{1}{2} - \ln(600-15e^{-\frac{1}{2}}) = c$ $\quad$ as $\quad 600-15e^{-\frac{1}{2}} > 0$

$\therefore\ -\ln(600-m) = \frac{t}{60} - \frac{1}{2} - \ln(600-15e^{-\frac{1}{2}})$ $\qquad (m < 600)$

$\frac{1}{2} + \ln\left(\frac{600-15e^{-\frac{1}{2}}}{600-m}\right) = \frac{t}{60}$

$t = 30 + 60\ln\left(\frac{600-15e^{-\frac{1}{2}}}{600-m}\right)$ or

$t = 60\left[\frac{1}{2} + \ln\left(\frac{600-15e^{-\frac{1}{2}}}{600-m}\right)\right]$

$t = 60\ln\left[\left(\frac{600-15e^{-\frac{1}{2}}}{600-m}\right)e^{\frac{1}{2}}\right]$ $\qquad \left(\frac{1}{2} = \ln e^{\frac{1}{2}}\right)$

$t = 60\ln\left(\frac{600e^{\frac{1}{2}}-15}{600-m}\right)$ #

☆ [many students assume $t=0,\ m=15$ which is disaster and get 0 m]

☆ [Some students use $t=0,\ m=15e^{-\frac{1}{2}}$, they need to refine t]

☆ [many students express $m = \cdots$ instead of $t=$ only gets max 1 m]

MATHEMATICS Extension 1 : Question 12		
Suggested Solutions	**Marks**	**Marker's Comments**
a) i) $x=\log_e t \rightarrow t=e^x$		
$\therefore y=t^2-2 \rightarrow y=e^{2x}-2$	1	
ii) $V=\pi\int_{\ln 2}^{\ln 4}(e^{2x}-2)^2\,dx$	1	Correct integral
$=\pi\int_{\ln 2}^{\ln 4} e^{4x}-4e^{2x}+4\,dx$		
$=\pi\left[\frac{e^{4x}}{4}-2e^{2x}+4x\right]_{\ln 2}^{\ln 4}$	1	Integrate
$=\pi\left[\frac{1}{4}(4)^4-2(4)^2+4\ln 4\right]-$		
$\pi\left[\frac{1}{4}(2)^4-2(2)^2+4\ln 2\right]$		
$=\pi(64-32+4\ln 4-4+8-4\ln 2)$		
$=\pi(36+8\ln 2-4\ln 2)$		
$=\pi(36+4\ln 2)\ u^3$	1	Answer

MATHEMATICS Extension 1 : Question 12		
Suggested Solutions	**Marks**	**Marker's Comments**
b) i $\frac{dN}{dt} = \frac{0.02N(8-N)}{8}$		
$\frac{dt}{dN} = \frac{8}{0.02N(8-N)}$		
$= 50 \times \frac{8}{N(8-N)}$		
$\therefore t = 50\int \frac{1}{N} + \frac{1}{8-N}\, dN$	1	using the identity given and setting up the integrals respectively.
$= 50\left[\ln\lvert N\rvert - \ln\lvert 8-N\rvert\right] + C$		
$= 50\ln\left\lvert\frac{N}{8-N}\right\rvert + C$		
when $t=0$, $N=7$		
$\therefore 50\ln 7 + C = 0$		
$\therefore C = -50\ln 7$	1	Working out the constant
$\therefore t = 50\ln\left\lvert\frac{N}{8-N}\right\rvert - 50\ln 7$		
$= 50\ln\left\lvert\frac{N}{56-7N}\right\rvert$		
$\frac{t}{50} = \ln\left\lvert\frac{N}{56-7N}\right\rvert$		
$\frac{N}{56-7N} = e^{\frac{t}{50}} \quad (N<8)$		
$N = 56e^{\frac{t}{50}} - 7Ne^{\frac{t}{50}}$	1	Make N the subject in terms of t only.

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
$N + 7Ne^{\frac{t}{50}} = 56e^{\frac{t}{50}}$		
$N\left(1+7e^{\frac{t}{50}}\right) = 56e^{\frac{t}{50}}$		
$N = \dfrac{56e^{\frac{t}{50}}}{1+7e^{\frac{t}{50}}}$	1	$N = \dfrac{56}{e^{\frac{-t}{50}}+7}$ is also acceptable
(ii) $N = \dfrac{56}{e^{\frac{-t}{50}}+7}$		
as $t \to \infty$, $e^{\frac{-t}{50}} \to 0$	1	1 mark for reasoning that limiting population means $t \to \infty$ or $\frac{dN}{dt} \to 0$, and attempting to arrive at an answer.
$\therefore N \to \dfrac{56}{7}$		
$= 8$		
$\therefore$ The limiting population is 8.	1	

MATHEMATICS Extension 1 : Question........		
Suggested Solutions	**Marks**	**Marker's Comments**

b)i Alternatively :

$$\frac{8\,dN}{0.02N(8-N)} = dt$$

$$50\int_7^N \frac{8}{N(8-N)}\,dN = \int_0^t dt$$ — 1

$$50\int_7^N \frac{1}{N} + \frac{1}{8-N}\,dN = \int_0^t dt$$

$$50\Big[\ln|N| - \ln|8-N|\Big]_7^N = t$$

$$\therefore \Big[50\ln\Big|\frac{N}{8-N}\Big|\Big]_7^N = t$$

$$50\ln\Big|\frac{N}{8-N}\Big| - \underline{50\ln 7} = t$$ — 1

This is the equivalent of working out the constant.

$$\ln\Big|\frac{N}{8-N}\Big| - \ln 7 = \frac{t}{50}$$

$$\ln\Big|\frac{N}{56-7N}\Big| = \frac{t}{50}$$

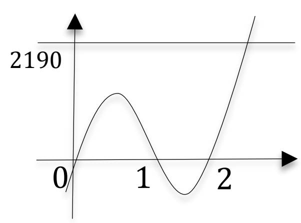
$\vdots$ Same as previous page

$$N = \frac{56e^{\frac{t}{50}}}{1+7e^{\frac{t}{50}}}$$ — 1

MATHEMATICS Extension 1 : Question........		
Alternate **Suggested Solutions**	**Marks**	**Marker's Comments**
b ii) $t = 50\int \frac{1}{N} + \frac{1}{8-N}\, dN$	1	
$\vdots$		
$50\ln\left\lvert\frac{N}{8-N}\right\rvert = t - c$		
$\ln\left\lvert\frac{N}{8-N}\right\rvert = \frac{t}{50} - \frac{c}{50}$		
$\frac{N}{8-N} = Ae^{\frac{t}{50}}$ where $A = e^{\frac{-c}{50}}$		
when $t=0$, $N=7$		
$\therefore A = 7$	1	
$\vdots$		
$N = \frac{56e^{\frac{t}{50}}}{1+7e^{\frac{t}{50}}}$	1	

MATHEMATICS Extension 1 : Question 13 - JRAHS - TASK 2 – TERM 1 - 2021		
Suggested Solutions	**Marks**	**Marker's Comments**
(a)		
$\frac{dy}{dx} = \frac{1}{2}\cos 2x \sin 3x$		
$y = \frac{1}{2}\int \sin 3x \cos 2x \, dx$	1	Separation of variables and integrating
$= \frac{1}{2}\int \frac{1}{2}(\sin(3x+2x) + \sin(3x-2x))dx$	1	Correctly applying the product to sum relationship
$= \frac{1}{4}\int (\sin 5x \ + \ \sin x) \, dx$		
$= \frac{1}{4}\left(-\frac{1}{5}\cos 5x \ - \ \cos x\right) + C$		
$= -\left(\frac{\cos 5x}{20} + \frac{\cos x}{4}\right) + C$	1	Correctly integrating, including constant of integration
ALTERNATE SOLUTION:		
$y = \frac{1}{2}\int \cos 2x \sin 3x \ dx$		NOTE: If you are not careful, you will land in hot water (many negatives to deal with)
$= \frac{1}{2}\int \frac{1}{2}(\sin(2x+3x) - \sin(2x-3x))dx$		
$= \frac{1}{4}\int (\sin 5x - \ \sin(-x)) \, dx$		$-\int \sin(-x)d(-x)$ $= -\cos(-x).(-1)$ $= \cos x$
$= \frac{1}{4}\int (\sin 5x + \sin x)dx$		
$= \frac{1}{4}\left(-\frac{1}{5}\cos 5x - \cos x\right) + C$		Look at the reference sheet and be familiar with data given (DO NOT RE-INVENT THE WHEEL!)
$= -\left(\frac{\cos 5x}{20} + \frac{\cos x}{4}\right) + C$		
$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$ $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$ $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$		

(b) Let the upper parabola be $y_1 = ax^2 - c$ and the lower parabola be $y_2 = bx^2 - (c+h)$ Points of intersection: $y_1 = y_2 \quad \Rightarrow \quad x^2(a-b) + h = 0$ When $x = \frac{w}{2}$, then $$(a-b) = \frac{-4h}{w^2}$$ Since $y_1 - y_2$ is an even function, $$A = 2\int_0^{\frac{w}{2}} (y_1 - y_2)dx$$ $$= 2\int_0^{\frac{w}{2}} (x^2(a-b) + h)dx$$ $$= 2\left[\frac{x^3}{3}(a-b) + hx\right]_0^{\frac{w}{2}}$$ $$= \frac{2}{3}\left(\frac{w^3}{8}\right)\left(\frac{-4h}{w^2}\right) + 2h\left(\frac{w}{2}\right) - 0$$ $$= -\frac{wh}{3} + wh$$ $$= \frac{2}{3}hw$$	1 1 1	For choosing the correct form of the parabolae together with h incorporated into it. For correctly obtaining the x-coordinate of the points of intersection (For use in the integral limits) For correctly integrating with correct algebraic manipulation. NOTE (i) Candidates who at the outset had the correct form of the parabola, were largely successful in getting to the final result. (ii) Those candidates who assumed that the graphs were drawn to scale, were awarded a maximum of 2 marks as the computation was trivial, and the question was made simpler.

(c)		
Suppose the woman has n friends. She has ${}^{n}C_3$ number of days to take a group of 3 friends. The least number of diners implies that ${}^{n}C_3 \geq 365$, for $n \geq 3$ by the pigeon hole principle. When $n = 4,$ this can be achieved in 4 days.	1	For showing some evidence that the PHP needed to be applied. (Entry Mark)
Clearly, when $n = 14$, then ${}^{14}C_3 = 364\ days$ which is insufficient. Hence when $n = 15$, then ${}^{15}C_3 = 455\ days$ which satisfies the requirement.	1	For some logical justification/reasoning
Thus, there must be at least 15 friends. NOTE: (a) This will still apply if a leap year is chosen i.e. 366 days (b) Solution to ${}^{n}C_3 \geq 365$ is not trivial! i.e. $\frac{n!}{3!(n-3)!} \geq 365$ $\Rightarrow \frac{n(n-1)(n-2)(n-3)!}{3!\,(n-3)!} \geq 365$ $\Rightarrow \frac{n(n-1)(n-2)}{3!} \geq 365$ i.e. $n(n-1)(n-2) \geq 2190$  Hence need to solve by inspection: Try $n = 3, 4, \ldots \ldots .10, \ldots . .14, 15, 16$	1	For final correct determination