Section I - (5 marks) Answer on the Multiple Choice answer sheet provided

1. $\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

   A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

   B. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

   C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

   D. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{8}\right)+C$

2. $\int \sec^4 x \tan x \, dx$ equals

   A. $\frac{1}{4}\sec^4 x + C$

   B. $\sec^4 x + C$

   C. $\frac{\sec^3 x}{3} + C$

   D. $\sec^3 x + C$

3. A solution to the differential equation $\frac{dy}{dx} = \sin(x+y) + \cos(x-y)$ can be obtained from

   A. $\int[\sin(y)+\cos(y)]\,dy = \int[\sin(x)+\cos(x)]\,dx$

   B. $\int \frac{1}{\sin(y)+\cos(y)}\,dy = \int[\sin(x)+\cos(x)]\,dx$

   C. $\int \frac{1}{\cos(y)-\sin(y)}\,dy = \int[\cos(x)-\sin(x)]\,dx$

   D. $\int \sec(y)\,dy = \int 2\sin(x)\,dx$

4. The differential equation $(\tan^{-1} x)^2 \frac{dy}{dx} - \frac{1}{1+x^2} = 0$ is equivalent to

   A. $t^2 \frac{dy}{dt} - 1 = 0$ where $t = \tan^{-1} x$

   B. $t^2 \frac{dy}{dt} - \frac{1}{1+t^2} = 0$ where $t = \tan^{-1} x$

   C. $\frac{dy}{dt} - \frac{1}{1+t^2} = 0$ where $t = \tan^{-1} x$

   D. $\frac{dy}{dt} - t^2 = 0$ where $t = \tan^{-1} x$

5. Which of the following is NOT equivalent to ${}^{n}C_{r}$?

   A. $\frac{n!}{(n-r)!r!}$

   B. ${}^{n}P_{r} \div r!$

   C. ${}^{n-1}C_{r} + {}^{n-1}C_{r+1}$

   D. ${}^{n+1}C_{r+1} - {}^{n}C_{r+1}$

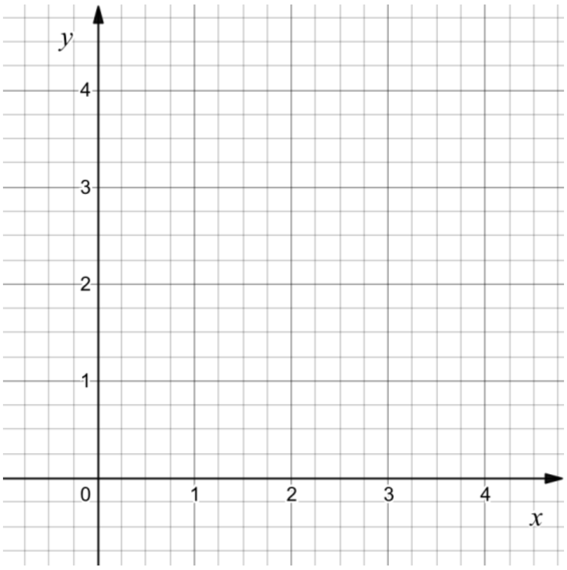
Section II – Answer on paper provided. Start a new page for each question.

Question 6 (9 marks) Start a new page **Marks**

a) Find $\int \cos 2x \cos 4x \, dx$ (2 marks)

b) Consider the differential equation $\frac{dy}{dx} + \frac{y}{x} = 0$.

(i) Construct a slope field for the differential equation using 1 unit interval for both x and y within the intervals $1 \leq x \leq 4$ and $1 \leq y \leq 4$. Draw each tangent line about 0.5 unit long. Use the grid provided on the reverse of the Multiple Choice answer sheet to draw your answer. (3 marks)



(ii) Sketch the solution curve to the differential equation $\frac{dy}{dx} + \frac{y}{x} = 0$ through $(2, 2)$. (1 mark)

c) A box contains 7 identical coloured discs of which 2 are green, 2 are yellow and 3 are red. If the discs are put in a line from left to right, find the total number of arrangements if:

(i) There are no restrictions. (1 mark)

(ii) A red disc cannot be at the left end of the line. (1 mark)

Question 7 (9 marks) Start a new page

a) Find $\int \frac{x^2}{\sqrt{1-x^2}} dx$ given $x = \sin\theta$. (3 marks)

b) Kevin is planning to invite his friends to a pool party. What is the number of friends that he must invite to guarantee that there are at least 3 friends born on the same day of the week? Explain your answer. (2 marks)

c) Solve $(x^2 + 1)\frac{dy}{dx} + 4xy = x$ and $y(0) = 2$. (4 marks)

Question 8 (8 marks) Start a new page **Marks**

a) Differentiate $y = \cos^{-1}\sqrt{\frac{1+x}{2}}$ (3 marks)

b) There are 5 children wishing to play "tag" in a garden in the shape of an equilateral triangle. The sides of the garden are all 20 metres long. Prove, by the pigeonhole principle, that there will always be at least 2 children within 10 metres of each other. (2 marks)

c) Find the largest interval on which the solution to the initial value problem $\frac{dy}{dx} = 1 + y^2$, where $y(0) = 0$, is defined. (3 marks)

Question 9 (8 marks) Start a new page

a) Find $\int \frac{dx}{x^2+6x+11}$ (2 marks)

b) A tank with vertical walls and rectangular cross-section is initially full of water. The water is leaking out of a hole in the base of the tank at a rate which is proportional to the square root of the depth of the water h. V m^3 is the volume of water in the tank at time t hours.

(i) Show that $\frac{dV}{dt} = a\sqrt{V}$, where a is a constant. (1 mark)

(ii) Given that the tank is half full after one hour, show that $V = V_0\left(\left(\frac{1}{\sqrt{2}} - 1\right)t + 1\right)^2$, where V_0 m^3 is the initial volume of water in the tank. (4 marks)

(iii) Hence show that the tank will be empty after approximately 3 hours and 25 minutes. Round your final answer to whole number. (1 mark)

Question 10 (8 marks) Start a new page

a) Find $\int x^2\sqrt{x-1}\ dx$ given $u = \sqrt{x-1}$ (3 marks)

b) A ten-member committee consist of 5 students, 3 teachers and 2 parents. The committee meets around in a circular table.

(i) How many different arrangements are possible if there are no restrictions? (1 mark)

(ii) How many different arrangements are possible if all the students sit together, as do the teachers, but no teacher sits next to a student? (2 marks)

(iii) One student and parent are related. If the condition in (ii) is satisfied, what is the probability that these two members sit next to each other? Explain your answer. (2 marks)

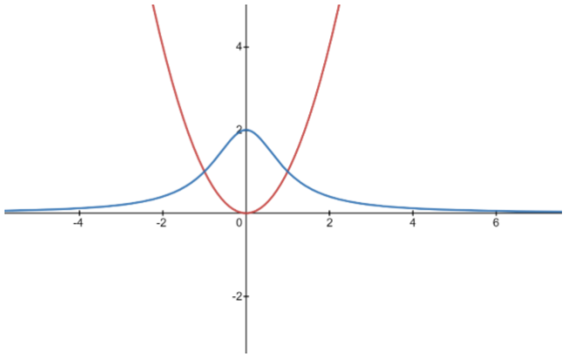
Question 11 (10 marks) Start a new page **Marks**

a) There are 8 players competing in a knock-out tennis tournament. The first round of matches is randomly drawn, with the players who lose the matches being knocked out of the competition immediately. The 4 winners of the first round will again be randomly assigned an opponent in the second round, with the winners advancing to the grand finale. The winner of the grand final is crowned the champion. There are no play-offs for minor places.

(i) How many ways can the first round of matches be drawn? (2 marks)

(ii) How many ways can the whole tournament be played out? You may assume that all players are equally skilled and have a 50% chance of winning each match. (2 marks)

b) The curve $y = x^2$ is a parabola and the curve $y = \frac{2}{1+x^2}$ is a bell-shaped curve. Find the area bounded by these curves. (3 marks)



c) The region S is bounded by the curve with equation $y = \frac{4}{x\left(4-\frac{1}{x}\right)^{\frac{1}{4}}}$, the x-axis and the lines with equations $x = 1$ and $x = \sqrt{2}$. The region S is rotated through 360° about the x-axis to form a solid of revolution. (3 marks)
Find the exact volume of the solid of revolution formed.

Question 12 (8 marks) Start a new page

A particle is moving in a straight line with velocity v ms^{-1}. At any time, t seconds, the velocity of the particle is given by the differential equation $2\frac{dv}{dt} + v^2 + 1 = 0$, where $0 \le t < a$.

(i) If $v = 1$ when $t = 0$, use calculus methods to show that $v = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$. (3 marks)

(ii) Given that $x = 0$ when $t = 0$, use calculus methods to find an expression for x, the displacement of the particle, in terms of t. (4 marks)

(iii) Find, correct to two decimal places, the position of the particle after $\frac{5\pi}{4}$ seconds. (1 mark)

Question 6

MC answers
1B 2A 3B 4A 5D

Tuesday, 15 March 2022 8:23 pm

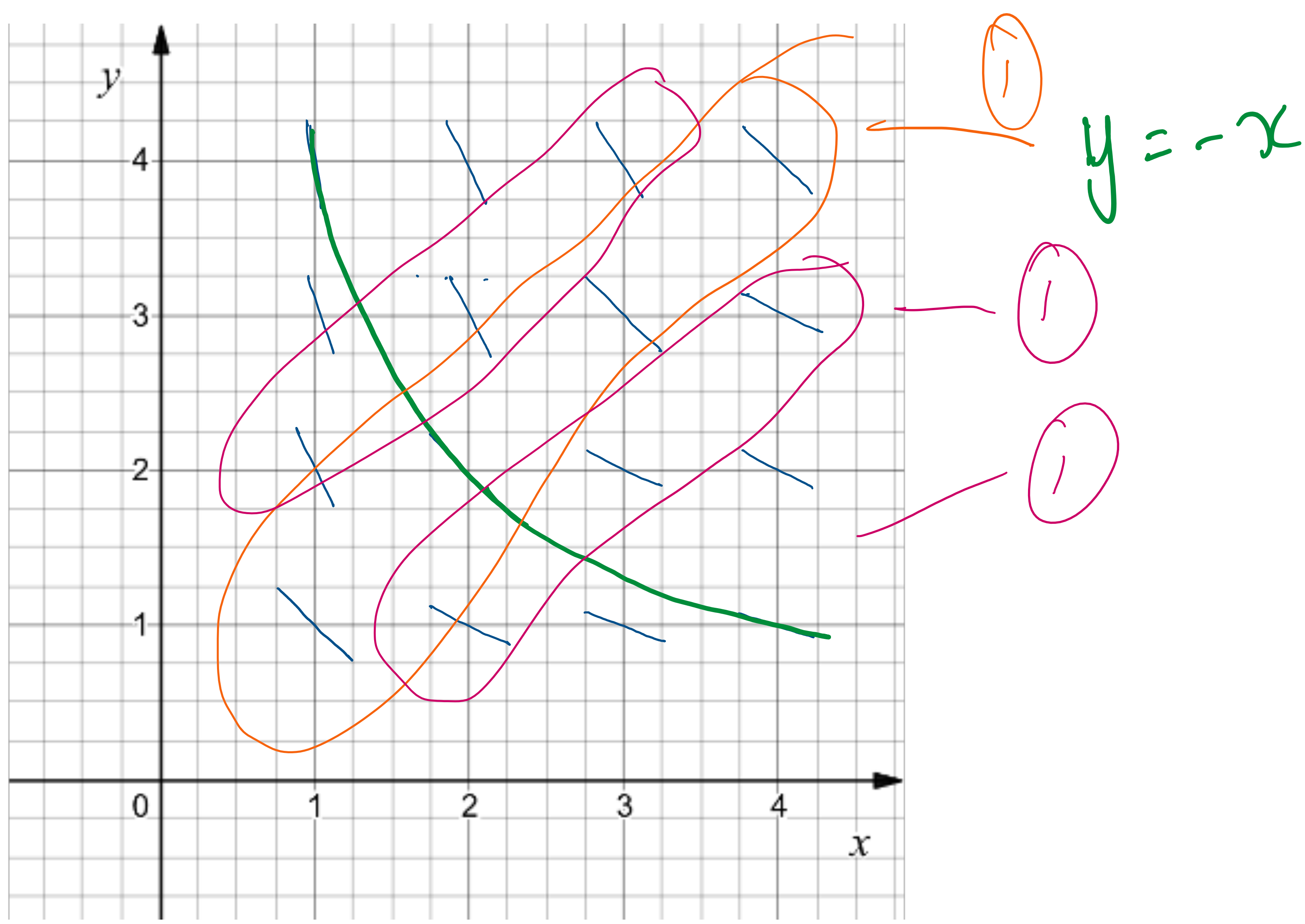
a) $\frac{1}{2}\int \left[\cos(4x-2x) + \cos(4x+2x)\right] dx$

$= \frac{1}{2}\int (\cos 2x + \cos 6x)\, dx$ 1

$= \frac{1}{2} \times \left(\frac{1}{2}\sin 2x + \frac{1}{6}\sin 6x\right) + C$

$= \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$ 1

b)



c) (i) $\frac{7!}{2!\,2!\,3!} = 210$ 1

(ii) $\frac{4 \times 6!}{2!\,2!\,3!} = 120$ 1

Question 7 (9 marks) Start a new page

a) Find $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ given $x = \sin\theta$. (3 marks)

$x = \sin\theta \therefore dx = \cos\theta\, d\theta$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta} \cos\theta\, d\theta$$

$$= \int \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int (1-\cos(2\theta))\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{2}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + C$$

b) Kevin is planning to invite his friends to a pool party. What is the number of friends that he must invite to guarantee that there are at least 3 friends born on the same day of the week? Explain your answer. (2 marks)

If each day of the week has 2 kids' birthday on it, that means we have 14 kids. By the 15th kid, there will be at least 3 kids born on the same day.

c) Solve $(x^2+1)\frac{dy}{dx} + 4xy = x$ and $y(0) = 2$. (4 marks)

$$(x^2+1)\frac{dy}{dx} + 4xy = x$$

$$(x^2+1)\frac{dy}{dx} = x(1-4y)$$

$$\int \frac{1}{1-4y}\,dy = \int \frac{x}{x^2+1}\,dx$$

$$-\frac{1}{4}\ln|1-4y| = \frac{1}{2}\ln(x^2+1) + C_1$$

$$\ln|1-4y| = -2\ln(x^2+1) + C_2$$

$$\ln|1-4y| = \ln(x^2+1)^{-2} + C_2$$

Now $y(0) = 2 \therefore$

$$\ln|1-4(2)| = \ln(0^2+1)^{-2} + C_2$$

$$\therefore C_2 = \ln 7$$

$$\ln|1-4y| = \ln(x^2+1)^{-2} + \ln 7$$

$$= \ln\left[7(x^2+1)^{-2}\right]$$

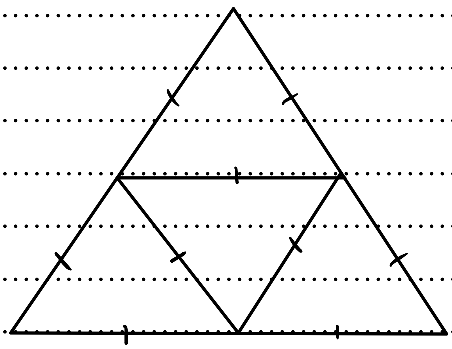
$$4y - 1 = 7(x^2+1)^{-2}$$

$$y = \frac{1 + 7(x^2+1)^{-2}}{4}.$$

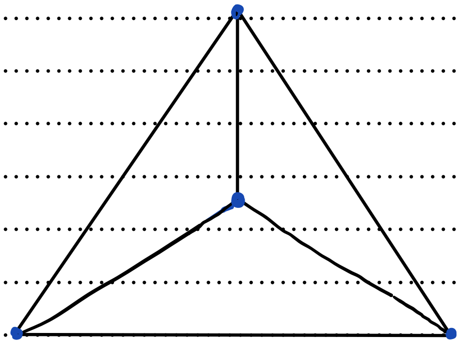
MATHEMATICS EXT 1: Question 8

Suggested Solutions	Marks	Marker's Comments
a, $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$		
$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\sqrt{\frac{1+x}{2}}\right)^2}} \times \frac{d}{dx}\left(\sqrt{\frac{1+x}{2}}\right)$		① Standard form for $\frac{d}{dx}(\cos^{-1} f(x))$
$= \frac{-1}{\sqrt{1-\frac{1+x}{2}}} \times \frac{1}{2}\left(\frac{1+x}{2}\right)^{-1/2} \times \frac{1}{2}$		① Inside derivative
$= \frac{-1}{\sqrt{\frac{2-(1+x)}{2}}} \times \frac{\sqrt{2}}{\sqrt{1+x}} \times \frac{1}{4}$		Common error: $1 - \frac{1+x}{2} = \frac{2-1+x}{2}$ $= \frac{1+x}{2}$
$= \frac{-\sqrt{2}}{\sqrt{1-x}} \times \frac{\sqrt{2}}{\sqrt{1+x}} \times \frac{1}{4}$		
$= \frac{-1}{2\sqrt{1-x^2}}$		① Simplifying (This was marked on a relative basis)
Note: Some students tried to rearrange to make x the subject and computed $\frac{dx}{dy}$. Almost all got it completely wrong.		

MATHEMATICS EXT 1: Question

Suggested Solutions	Marks	Marker's Comments
b) 		
Divide the garden into 4 smaller regions shaped as equilateral triangles, each with side length 10 m. (These regions play the role of the pigeonholes).		
We also have 5 children to be placed in the garden. (These children play the role of the pigeons).		
By the Pigeonhole Principle, there is at least one smaller region with at least two children within it.	1	Don't even have to mention PHP, so long as concept is explained.
Since the furthest distance two people can be in an equilateral triangle is if they are at two verticies, then the maximum distance between the two children in the same region is 10m.	1	You MUST have an explanation for why the two students are 10m apart. Simply stating it is fudging.
$\therefore$ There are always two students within 10m of each other.		

MATHEMATICS EXT 1: Question

Suggested Solutions	Marks	Marker's Comments
 Note: Students who argued with a diagram like this mostly got 1 or 0 marks. ↳ 1 mark for correctly placing children in "worse case" ↳ 1 mark for a correct explanation.		

MATHEMATICS EXT 1: Question

Suggested Solutions	Marks	Marker's Comments
c, $\frac{dy}{dx} = 1+y^2$		
$\frac{dy}{1+y^2} = dx$		Common error:
		$\int \frac{1}{1+y^2}\,dy = \tan(y)$
$\int \frac{dy}{1+y^2} = \int dx$		$\int \frac{1}{1+y^2}\,dy = \ln\|1+y^2\|$
$\tan^{-1}(y) = x + C$	1	
$y(0)=0 \Rightarrow \tan^{-1}(0) = 0 + C$		
$C=0$	1	
$\therefore \tan^{-1}(y) = x \quad \Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$		
$y = \tan(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$	1	

Year 11 Extension 1: Q9		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
(a) $$I=\int\frac{dx}{x^2+6x+4}=\int\frac{dx}{(x+3)^2+2}=\frac{1}{2}\int\frac{dx}{\left(\frac{x+3}{\sqrt{2}}\right)^2+1}$$ $$u=\frac{1}{\sqrt{2}}(x+3)\to du=\frac{1}{\sqrt{2}}dx$$ so $$I=\frac{\sqrt{2}}{2}\int\frac{du}{u^2+1}$$ $$=\frac{1}{\sqrt{2}}\tan^{-1}(u)+C$$ $$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C$$	1 Completing the square 1 Final result	Not many issues in this question.
(b) (i) $$\frac{dV}{dt}\propto-\sqrt{h}$$ $V=Ah$ where A is constant (base/cross-sectional area). So $$h=\frac{V}{A}\to\frac{dV}{dt}=-k\sqrt{\frac{V}{A}}\equiv a\sqrt{V}$$ where $a=-\frac{k}{\sqrt{A}}$, $k>0$, a constant.	1	Needed to link h to V and hence show why $\frac{dV}{dt}=a\sqrt{V}$. Many had problems doing this.
(ii) $$\frac{dV}{dt}=a\sqrt{V}$$ so, by separation of variables, $$\int\frac{dV}{\sqrt{V}}=\int a\,dt$$ $$2\sqrt{V}=at+C$$ Boundary condition: $t=0, V=V_0$, so $$2\sqrt{V_0}=C$$ so	1 First integral 1 Evaluation of constant of integration	

$$2\sqrt{V} = at + 2\sqrt{V_0}$$ Also, when $t = 1, V = V_0/2$, so $$2\sqrt{\frac{V_0}{2}} = a + 2\sqrt{V_0}$$ so $$a = \sqrt{V_0}(\sqrt{2} - 2)$$ hence $$2\sqrt{V} = \left(\sqrt{V_0}(\sqrt{2} - 2)\right)t + 2\sqrt{V_0}$$ $$2\sqrt{V} = \sqrt{V_0}\left((\sqrt{2} - 2)t + 2\right)$$ $$\sqrt{V} = \sqrt{V_0}\left(\left(\frac{1}{\sqrt{2}} - 1\right)t + 1\right)$$ $$\therefore V = V_0\left(\left(\frac{1}{\sqrt{2}} - 1\right)t + 1\right)^2$$	1 Evaluation of constant, a 1 Final result	
(iii) Time taken to empty, set $V = 0$: $$0 = V_0\left(\left(\frac{1}{\sqrt{2}} - 1\right)t + 1\right)^2$$ $$t = \frac{1}{1 - \frac{1}{\sqrt{2}}} = 2 + \sqrt{2} \approx 3 \text{ hours}, 25 \text{ minutes}$$	1	A lot of students established some estimate for time taken to empty (41/12) and then substituted to show that this gave $V \approx 0$. Problem: from where did that number come? Also, the question expects the exact answer first, then an approximation in hours and minutes. There was no award if handled like this since it's technically incorrect and there is only one mark available.

Year 12 Task 2 MATHEMATICS Ext1 Question 10

Suggested Solutions	Marks	Marker's Comments

a) let $I = \int x^2\sqrt{x-1}\,dx$

let $u = \sqrt{x-1}$ and $x^2 = (u^2+1)^2$

$\therefore \frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x-1}}$ — (1)

$= \frac{1}{2u}$

$\therefore dx = 2u\,du$

$I = \int (u^2+1)^2 u \times 2u\,du$

$= \int 2u^2(u^4+2u^2+1)\,du$

$= \int (2u^6+4u^4+2u^2)\,du$ — (1)

$= \left[\frac{2u^7}{7} + \frac{4u^5}{5} + \frac{2u^3}{3}\right] + C$

$= \frac{2(x-1)^3\sqrt{x-1}}{7} + \frac{4(x-1)^2\sqrt{x-1}}{5} + \frac{2(x-1)\sqrt{x-1}}{3} + C$ — (1)

Year 12 Task 2 MATHEMATICS Ext 1 Question 10

Suggested Solutions	Marks	Marker's Comments

b)

i There are 10 people arranged in a circle.

$\therefore$ Total $= (10-1)!$

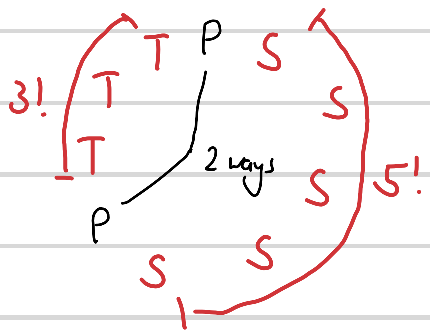
$= 9!$

$= 362880$ — 1

ii - First the 5 students sit down in $5!$ ways — 1 for answer

- Then the teachers in $3!$ ways

- To separate the teachers and students, place a parent on either end of the students - 2 ways — 1 Explanation or working out



$\therefore$ Total $= 5! \times 3! \times 2$

$= 1440$

iii Once the parent has sat down, the student only has 5 positions to sit in. — 1 for answer

$\therefore$ Probability $= \frac{1}{5}$ — 1 Explanation or working out

2022 Task 2
Mathematics Extension 1

Solutions – Question 11

π

Question 11, Part A

There are 8 players competing in a knock-out tennis tournament. The first round of matches is randomly drawn, with the players who lose the matches being knocked out of the competition immediately. The 4 winners of the first round will again be randomly assigned an opponent in the second round, with the winners advancing to the grand finale. The winner of the grand final is crowned the champion. There are no play-offs for minor places.

(i) How many ways can the first round of matches be drawn?

(ii) How many ways can the whole tournament be played out? You may assume that all players are equally skilled and have a 50% chance of winning each match.

Answers, 11 a)

(i) 1

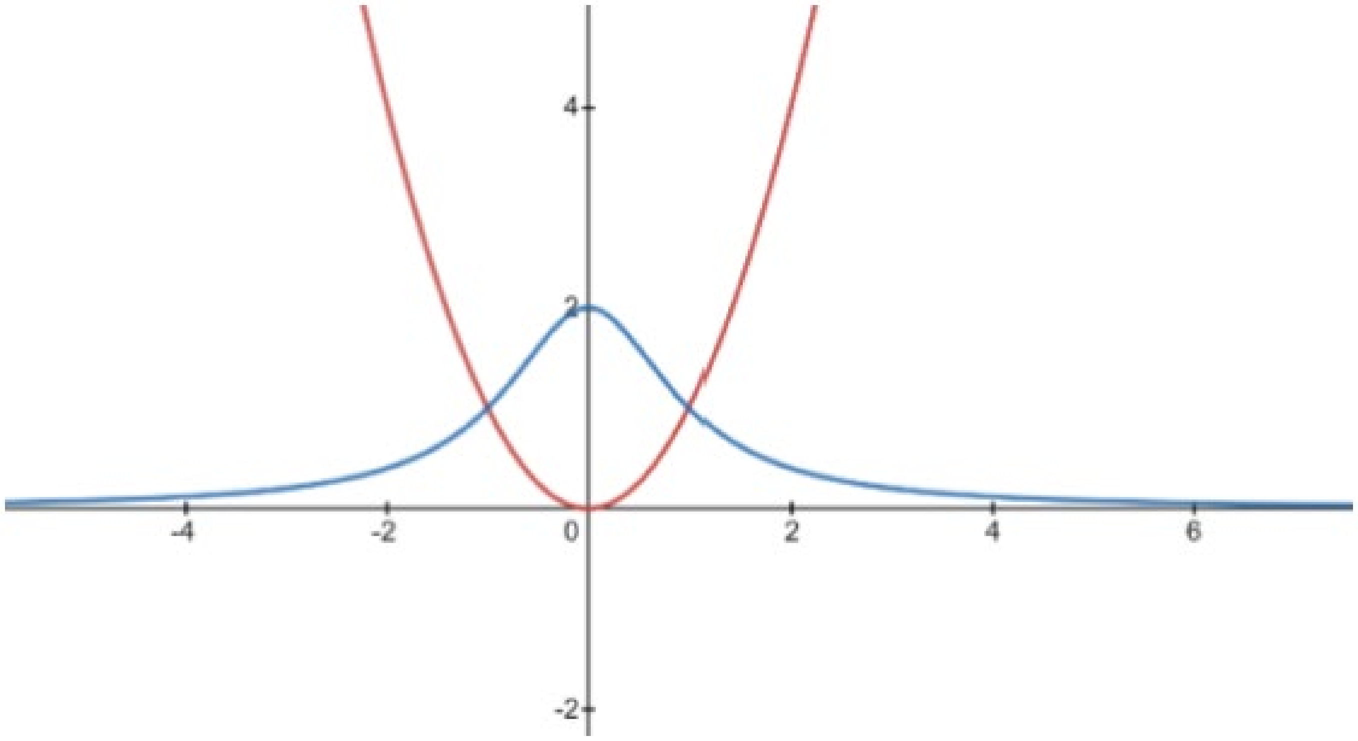
$$\frac{{}^8_2C \times {}^6_2C \times {}^4_2C \times {}^2_2C}{4!} = 105$$ 1

(ii) $$105 \times \frac{{}^4_2C \times {}^2_2C}{2!} \times 2 \times 2^4 \times 2^2 = 40320$$

1 1

QUESTION 11, PART B

The curve $y = x^2$ is a parabola and the curve $y = \frac{2}{1+x^2}$ is a bell-shaped curve. Find the area bounded by these curves.



Answer, 11 b)

$$y = x^2$$

$$y = \frac{2}{1+x^2}$$

$$x^2 = \frac{2}{1+x^2}$$

$$x^4 + x^2 - 2 = 0$$

$$x^2(x^2+2) - 1(x^2+2) = 0$$

$$(x^2-1)(x^2+2) = 0$$

$$x^2 = 1$$

1 So point of intersections are $(1,1)$ $(-1,1)$

Continued…………

$$\text{Required Area} = 2\left(\int_0^1 \frac{2}{x^2+1}\,dx - \int_0^1 x^2\,dx\right)$$

$$= 2\left(\left|2\tan^{-1}x\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1\right)$$ 1

$$= 2\left(\frac{\pi}{2} - \frac{1}{3}\right)$$

$$= \left(\pi - \frac{2}{3}\right)\ sq\ units.$$ 1

Question 11, Part C

The region S is bounded by the curve with equation $y = \frac{4}{x(4-\frac{1}{x})^{\frac{1}{4}}}$, the x-axis and the lines with equations $x = 1$ and $x = \sqrt{2}$. The region S is rotated through $360°$ about the $x - axis$ to form a solid of revolution. Find the exact volume of the solid of revolution formed.

Answer, 11 c)

$$V = \pi \int_1^{\sqrt{2}} y^2 dx$$

$$V = \pi \int_1^{\sqrt{2}} \left(\frac{4}{x((4-\frac{1}{x})^{\frac{1}{4}}}\right)^2 dx$$

$$V = \pi \int_1^{\sqrt{2}} \frac{16}{x^2(\sqrt{(4-\frac{1}{x}})} dx$$

1

$$sub\ u = 4 - \frac{1}{x}$$

$$du = \frac{1}{x^2} dx$$

$$V = 16\pi \int_3^{4-\frac{1}{\sqrt{2}}} u^{-\frac{1}{\sqrt{2}}} du$$

$$V = 16\pi \left|\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right|_3^{4-\frac{1}{\sqrt{2}}}$$

1

$$V = 32\pi\left(\sqrt{4-\frac{1}{\sqrt{2}}} - \sqrt{3}\right) \textit{ cubic units.}$$

1

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments
i) $2\frac{dv}{dt} + v^2 + 1 = 0$		
$\frac{dv}{dt} = \frac{-v^2-1}{2}$		
$\int \frac{dv}{v^2+1} = \int -\frac{1}{2}\,dt$	1	
$\tan^{-1} v = -\frac{1}{2}t + c$	1	
when $v=1,\ t=0$		
$\tan^{-1}(1) = 0 + c$		
$\therefore c = \frac{\pi}{4}$	1	
$\therefore \tan^{-1} v = -\frac{1}{2}t + \frac{\pi}{4}$		
$\therefore v = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$		
ii) $\frac{dx}{dt} = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$		
$\int dx = \int \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)dt$	1	
$x = -2\int \frac{-\frac{1}{2}\sin\left(\frac{\pi}{4} - \frac{t}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{t}{2}\right)}\,dt$		
$= 2\ln\left\lvert\cos\left(\frac{\pi}{4} - \frac{t}{2}\right)\right\rvert + c$	1	
when $x=0,\ t=0$		
$0 = 2\ln\left\lvert\cos\left(\frac{\pi}{4}\right)\right\rvert + c$		
$\therefore c = -2\ln\frac{1}{\sqrt{2}}$		
$= \ln 2$	1	
$\therefore x = 2\ln\left\lvert\cos\left(\frac{\pi}{4} - \frac{t}{2}\right)\right\rvert + \ln 2$	1	must have absolute values

MATHEMATICS Extension 1 : Question ..12..		
Suggested Solutions	**Marks**	**Marker's Comments**
iii) when $t = \frac{5\pi}{4}$ $x = 2\ln\left\|\cos\left(\frac{\frac{\pi}{4} - \frac{5\pi}{4}}{2}\right)\right\| + \ln 2$ $= -1.22794\ldots$ $= -1.23$ (2dp) $\therefore$ 1.23m to the left of the origin	1	if calc. was in degrees $x = 0.6927\ldots$