Student Name: ____________________

Maths class: ____________________

James Ruse Agricultural High School

2024 Year 12 Task 2

Mathematics Extension 1

General Instructions	**Total Marks: 55**
• Reading time – 5 minutes • Working time – 1 hour and 30 minutes • Write using black pen • NESA approved calculators & templates may be used • A reference sheet is provided • In Questions 6-12, show all relevant mathematical reasoning and/or calculations. • Marks may not be awarded for careless or badly arranged working.	**Section I – 5 marks** (pages 2-3) • Attempt Questions 1–5 • Answer on the Multiple Choice answer sheet provided. • Allow about 10 minutes for this section **Section II – 50 marks** (pages 4–7) • Attempt Questions 6–12 • Answer on lined paper provided. Start a new page for each new question • Allow about 1 hour 20 minutes for this section

The answers to all questions are to be returned in separate stapled bundles clearly labelled Question 6, Question 7, etc. Each question must show your Candidate Number.

Section I

5 Marks
Attempt Questions 1 to 5
Allow approximately 10 minutes for this section.

Answer on the Multiple Choice sheet provided.

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1. Which of the following is the derivative of $\cos^{-1}\frac{3}{x}$? **1**

   (a) $\frac{3}{\sqrt{x^2-9}}$

   (b) $\frac{-3}{\sqrt{x^2-9}}$

   (c) $\frac{3}{x\sqrt{x^2-9}}$

   (d) $\frac{-3}{x\sqrt{x^2-9}}$

2. If there are 4 girls and 4 boys, how many ways can they be arranged in a line if boys and girls are alternating, and Faith (girl) must stand at the front or back of the line? **1**

   (a) 144

   (b) 288

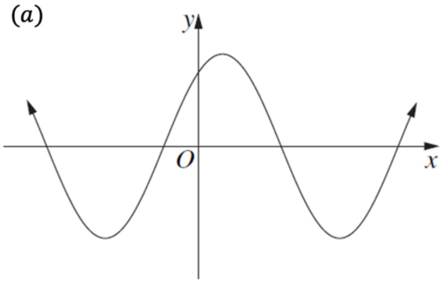
   (c) 576

   (d) 40320

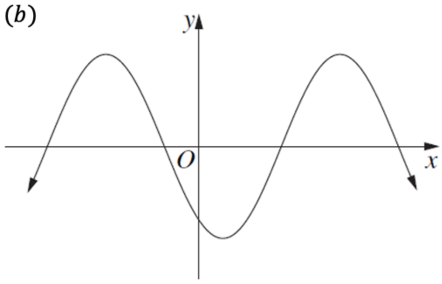
3. Which of the following curves best represents the graph of the function **1**

   $f(x) = A\sin x + B\cos x$, where both $A, B > 0$

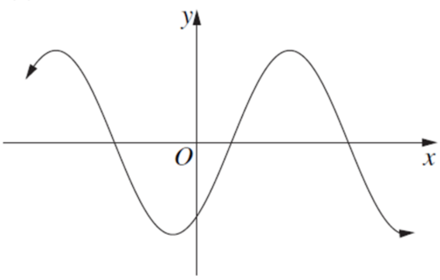
(a)



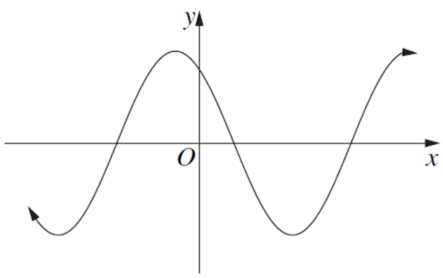
(b)



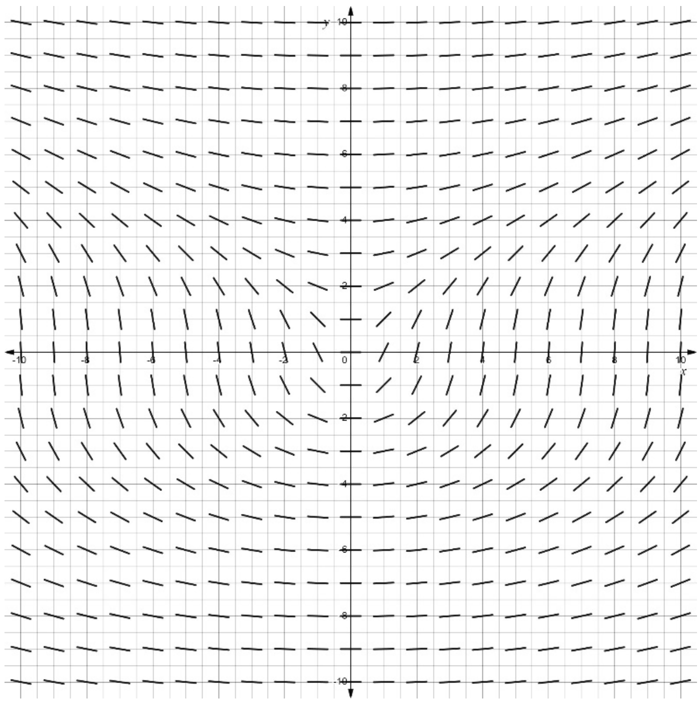
(c)



(d)



4. The slope field for a first order differential equation is shown. **1**



Which of the following differential equations best represent the slope field above?

(a) $\frac{dy}{dx} = \frac{2x}{y^2}$

(b) $\frac{dy}{dx} = \frac{2x^2}{y^2}$

(c) $\frac{dy}{dx} = \frac{2x}{y^2+1}$

(d) $\frac{dy}{dx} = \frac{2x^2}{y^2+1}$

5. Find the initial velocity if velocity is given by $v = 2e^{3t} - 5\cos\left(t - \frac{\pi}{3}\right)$ **1**

(a) 0.5 m/s

(b) 3 m/s

(c) -0.5 m/s

(d) -3 m/s

End of Section I

Section II

50 Marks
Attempt Questions 6 to 12
Allow approximately 1 hour 20 minutes for this section.

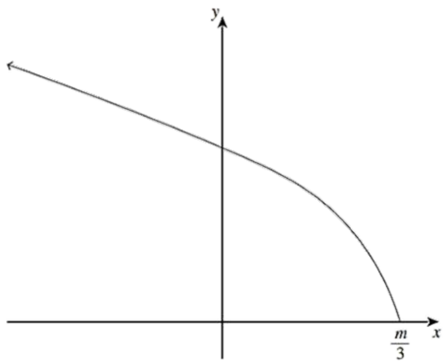
Write your answers on the paper supplied.
Start a new page for each new question.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (7 Marks)

(a) If 5 people are to be elected for a committee from a ballot of 4 women and 5 men, how many ways can this committee of 5 be chosen if there are to be at least 2 women selected? **2**

(b) Use the substitution $u = 1 + \ln x$ to find $\int \frac{\ln x}{x(1+\ln x)^2}\, dx$ **2**

(c) The graph of $f(x) = \sqrt{m - 3x}$ for $x \leq \frac{m}{3}$ is shown below.



The area enclosed by $f(x)$, the x-axis and the y-axis is rotated about the y-axis. Find the value of m such that the volume of the solid formed is $\frac{5000\pi}{27}$ units cubed. **3**

End of Question 6

Question 7 (7 Marks)

(a) Find $\int \sin^2 3x \, dx$ **2**

(b) Find $\int_0^{\frac{4}{3}} \frac{dx}{16+9x^2}$ **2**

(c) Show by Mathematical Induction that **3**

$$\sum_{k=1}^{n} \frac{k2^k}{(k+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

For all natural numbers $n \geq 1$.

End of Question 7

Question 8 (7 Marks)

(a) Find the particular solution to $\frac{dy}{dx} = \frac{6x-5}{4y}$ given that $y = 1$ at $x = 1$. **2**

(b) The displacement of a car from home is given by $x = -t^2 + 3t + 6$ where t is in seconds.

i. Find the velocity and acceleration of the car after 2 seconds. **1**

ii. Find when it changes direction. **1**

iii. Find the distance travelled in the first 2 seconds. **3**

End of Question 8

Question 9 (7 Marks)

(a) How many ways can 9 chocolates be shared amongst 2 people if the chocolates are identical? **1**

(b) 50 hawks were introduced to an island on the 1st of March 2024 to control the population of 5000 mice. The number of mice, M, started to decrease at a rate of $\frac{dM}{dt} = -k(M-25)$ while the population of hawks, H, was increasing at a rate of $\frac{dH}{dt} = 0.001H(250-H)$ where t is the time in years.

i. Given that $M = 25 + 4875e^{(-kt)}$ satisfies the differential equation, $\frac{dM}{dt} = -k(M-25)$, and there were 3821 mice left on the island after one year, show that $k = 0.25$ correct to two decimal places. **1**

ii. Given that $\frac{1}{0.001H(250-H)} = 4\left(\frac{1}{H} + \frac{1}{250-H}\right)$, show that the equation of the population of hawks is given by $H = \frac{250}{1+4e^{-0.25t}}$. **3**

iii. By using part a and b, in what year will the population of hawks exceed the population of mice? **2**

End of Question 9

Question 10 (7 Marks)

(a) Using a t-substitution where $t = \tan\theta$, show that: **1**

i.

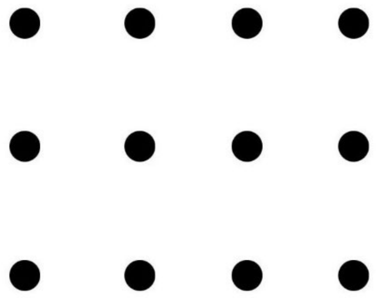
$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

ii. Using the result from part (a) i. or otherwise, show that: **2**

$$\tan\frac{\pi}{12} = 2-\sqrt{3}$$

(b) How many ways can the letters of the word JAMESRUSE be arranged so that the 2 E's are not together? **2**

(c) Find the number of triangles that can be formed in a 4x3 array of dots if the vertices of the triangle must lie on a dot. **2**



End of Question 10

Question 11 (7 Marks)

(a) i. Find the derivative of: $x \ln x - x$ **1**

iii. Using your result from part (a) i., evaluate $\int_{\sqrt{e}}^{e} \ln x \, dx$ **2**

(b) i. Express $\cos x - \sin x$ in the form $R\sin(x + \alpha)$, where α is in radians. **2**

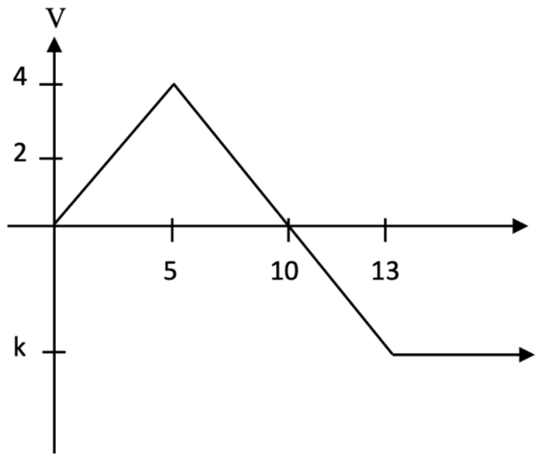
ii. Hence using your result from (b) i., or otherwise, sketch the graph of:

$$y = \cos x - \sin x \text{ for } 0 \leq x \leq 2\pi.$$ **2**

End of Question 11

Question 12 (8 Marks)

(a) Consider the velocity time graph below where velocity is in metres per second and time is in seconds. After 13 seconds, the car remains at a constant velocity k.



NOT TO SCALE

i. Find the value of k **1**

ii. At what time does the car return to its original position? **1**

(b) Using sum-to-product, find the solution to **3**

$$\cos 7x + \cos 5x = \cos 6x$$

for $0 \leq x \leq 2\pi$

(c) Show by Mathematical Induction that $2^{3^n} + 1$ is divisible by 3^{n+1} for all integers $n \geq 0$. **3**

End of Question 12

End of Section II

End of paper.

MATHEMATICS Extension 1 : Question…6…

Suggested Solutions	Marks	Marker's Comments
a) no. of ways = total − 0 women − 1 women		
$= {}^9C_5 - {}^5C_5 - {}^5C_4 \times {}^4C_1$	1	need to have the correct reasoning for 1 mark
$= 105$	1	
OR		$4C_2 \times {}^7C_3$ has many repeats.
no. of ways = 2 women + 3 women + 4 women		
$= {}^4C_2 \times {}^5C_3 + {}^4C_3 \times {}^5C_2 + {}^4C_4 \times {}^5C_1$	1	
$= 105$	1	
b) $\int \frac{\ln x}{x(1+\ln x)^2}\,dx$ $u = 1+\ln x$, $\ln x = u-1$; $\frac{du}{dx} = \frac{1}{x}$; $du = \frac{1}{x}dx$		
$= \int \frac{u-1}{u^2}\,du$	1	
$= \int \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du$		
$= \int \left(\frac{1}{u} - u^{-2}\right) du$		
$= \ln\lvert u\rvert + u^{-1} + c$		lots of students did not sub. x back.
$= \ln\lvert 1+\ln x\rvert + (1+\ln x)^{-1} + c$	1	
c) $y = \sqrt{m-3x}$ when $x=0$, $y = \sqrt{m}$		
$y^2 = m - 3x$		
$3x = m - y^2$		
$x = \frac{m-y^2}{3}$		
$V = \pi \int_0^{\sqrt{m}} \left(\frac{m-y^2}{3}\right)^2 dy$	1	If rotated about x-axis, ie. $V = \pi \int_0^{\frac{m}{3}} (\sqrt{m-3x})^2\,dx$ maximum is 1 mark if correct answer of $m = \frac{100}{3}$ is given
$= \frac{\pi}{9} \int_0^{\sqrt{m}} (m^2 - 2y^2m + y^4)\,dy$		
$= \frac{\pi}{9}\left[m^2y - \frac{2}{3}my^3 + \frac{1}{5}y^5\right]_0^{\sqrt{m}}$		
$= \frac{\pi}{9}\left[m^2\sqrt{m} - \frac{2}{3}m(\sqrt{m})^3 + \frac{1}{5}(\sqrt{m})^5 - 0\right]$		

MATHEMATICS Extension 1 : Question 6

Suggested Solutions	Marks	Marker's Comments
$\dfrac{5000\pi}{27} = \dfrac{\pi}{9}\left(m^{\frac{5}{2}} - \dfrac{2}{3}m^{\frac{5}{2}} + \dfrac{1}{5}m^{\frac{5}{2}}\right)$	1	
$\dfrac{8}{15}m^{\frac{5}{2}} = \dfrac{5000}{3}$		
$m^{\frac{5}{2}} = 3125$		
$\therefore m = 25$	1	

2024 - Yr 12 - Task 2 - Ext 1 - Q7

MATHEMATICS Extension 1 : Question 7

Suggested Solutions	Marks	Marker's Comments
a) $\int \sin^2 3x \, dx$ — applying that it has to be changed to degree 1 - reference sheet		
$= \frac{1}{2}\int (1-\cos 6x)\, dx$ — 1		
$= \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C$ or $\frac{x}{2} - \frac{\sin 6x}{12} + C$ — 1		
b) $\int_0^{4/3} \frac{dx}{16+9x^2}$ Again a reference sheet formula No !! Sympathy mark !		
M1 $\frac{1}{3}\int_0^{4/3} \frac{3\,dx}{4^2+(3x)^2}$		
$= \frac{1}{12}\left[\tan^{-1}\frac{3x}{4}\right]_0^{4/3}$ — 1		
$= \frac{1}{12}\left[\tan^{-1}\frac{3}{4}\left(\frac{4}{3}\right) - \tan^{-1}(0)\right]$		
$= \frac{1}{12}\left(\frac{\pi}{4}\right)$		
$= \frac{\pi}{48}$ or 0.065 — 1		
M2 - my prefered method.		
$\frac{dx}{\frac{16}{9}+x^2}$ $= \frac{1}{9}\int_0^{4/3} \frac{dx}{\left(\frac{4}{3}\right)^2+x^2}$	— 1	
$= \frac{1}{9}\left[\frac{1}{\frac{4}{3}}\tan^{-1}\left(\frac{x}{\frac{4}{3}}\right)\right]_0^{4/3}$		
$= \frac{1}{9}\times\left(\frac{3}{4}\tan^{-1}\left(\frac{4/3}{4/3}\right) - \frac{3}{4}\tan^{-1}\left(\frac{0}{4/3}\right)\right)$		— 1

$= \pi/48$

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments

c) Show $\sum_{k=1}^{n} \frac{k2^k}{(k+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$ for all $n \geq 1$

M1

Prove true for $n = 1$

LHS $\frac{1(2)^1}{(1+2)!} = \frac{1}{3}$

RHS $1 - \frac{2^{1+1}}{(1+2)!} = 1 - \frac{4}{6} = \frac{1}{3}$

LHS = RHS $\therefore$ true for $n = 1$ — 1

- State prove true for $n = 1$
- Show LHS & RHS separately & state true — 1

Some of you got LUCKY!! Your setting out is NOT ACCEPTABLE

Assume true for $n = m$ where $(m \in \mathbb{Z})$

$$\boxed{\sum_{k=1}^{m} \frac{k2^k}{(k+2)!} = 1 - \frac{2^{m+1}}{(m+2)!}}$$

Prove true for $n = m+1$

$$\sum_{k=1}^{m} \frac{k \cdot 2^k}{(k+2)!} + \frac{(m+1)2^{m+1}}{(m+3)!} = 1 - \frac{2^{m+2}}{(m+3)!}$$

must state this line to get the next mark & show the next lines of working out! ONLY Then the LHS statement will make sense to show (by assumption)!

LHS

$$\sum_{k=1}^{m} \frac{k \cdot 2^k}{(k+2)!} + \frac{(m+1)2^{m+1}}{(m+3)!}$$

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
$= 1 - \frac{2^{m+1}}{(m+2)!} + \frac{(m+1)2^{m+1}}{(m+3)!}$	(by	assumption)

common denominator

$$= 1 - \frac{2^{m+1}(m+3) - (m+1)2^{m+1}}{(m+3)!}$$

$$= 1 - \frac{2^{m+1}\left((m+3) - (m+1)\right)}{(m+3)!}$$

$$= 1 - \frac{2^{m+1} \cdot 2}{(m+3)!}$$

$$= 1 - \frac{2^{m+2}}{(m+3)!}$$

$= RHS$

→ If got $\frac{1 - 2^{m+1} \cdot 2(k+2)}{(k+3)!}$ → NOT TRUE! No marks

→ if got $1 - \frac{2^{m+1}(m+3-m-1)}{(m+3)!}$ → this alone, NO! mark allocated.

Have to show $1 - \frac{2^{m+2}}{(m+3)!}$

By the process of MI, this statement is true for $n = 1$ + $n = k+1$, $n \in \mathbb{Z}$

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments

M2

$$\sum_{k}^{n} \frac{k \cdot 2^k}{(k+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

$$\frac{1 \cdot 2^1}{(1+2)!} + \frac{2 \cdot 2^2}{(2+2)!} + \frac{3 \cdot 2^3}{(3+2)!} + \cdots + \frac{n \cdot 2^n}{(n+2)!}$$

showing this expansion — 1

Assume true for $n = k$ $(k \in Z^+)$

$$\frac{2}{(1+2)!} + \frac{2 \cdot 2^2}{(2+2)!} + \frac{3 \cdot 2^3}{(3+2)!} + \cdots + \frac{k \cdot 2^k}{(k+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$$

Proof true for $n = k+1$

$$\frac{1 \cdot 2^1}{(1+2)!} + \frac{2 \cdot 2^2}{(2+2)!} + \cdots + \frac{k \cdot 2^k}{(k+2)!} + \frac{(k+1) \cdot 2^{k+1}}{(k+1+2)!} = 1 - \frac{2^{k+1+1}}{(k+1+2)!}$$

$$= \frac{2}{(1+2)!} + \frac{2 \cdot 2^2}{(2+2)!} + \cdots + \frac{k \cdot 2^k}{(k+2)!} + \frac{(k+1) \cdot 2^{k+1}}{(k+3)!} = 1 - \frac{2^{k+2}}{(k+3)!}$$

LHS

$$1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1)2^{k+1}}{(k+3)!}$$

(by assumption) — 1

$$= 1 - \left[\frac{2^{k+1}((k+3) + (k+1) \cdot 2^{k+1}}{(k+3)!}\right]$$

$$= 1 - \frac{2^{k+1}((k+3) - (k+1))}{(k+3)!}$$

$$= \frac{1 - 2^{k+1} \cdot 2}{(k+3)!}$$

$$= 1 - \frac{2^{k+2}}{(k+3)!} = RHS$$

Conclusion as in M1 — 1

Extension 1: Question 8

<table>
<tr><th>Suggested Solutions</th><th>Marks</th><th>Marker's Comments</th></tr>
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(a) We have the differential equation with boundary condition:

$$\frac{dy}{dx} = \frac{6x-5}{4y};\ y(1) = 1$$

The DE is variables-separable since it may be decomposed into a product of two, single-variable functions in each of the variables x and y:

$$\frac{6x-5}{4y} = (6x-5)\cdot\frac{1}{4y}$$

Hence

$$\int 4y\ dy = \int 6x - 5\ dx$$

so

$$\frac{4y^2}{2} = \frac{6x^2}{2} - 5x + C \Longrightarrow 2y^2 = 3x^2 - 5x + C$$

Since $y = 1$ where $x = 1$, we have

$$2(1)^2 = 3(1)^2 - 5(1) + C \Longrightarrow C = 4$$

Hence

$$y^2 = \frac{1}{2}(3x^2 - 5x + 4)$$

Alternatively,

$$\int_1^y 4y\ dy = \int_1^x 6x - 5\ dx$$

so

$$2y^2|_1^y = (3x^2 - 5x)|_1^x$$

$$2y^2 - 2(1)^2 = (3x^2 - 5x) - \left(3(1)^2 - 5(1)\right)$$

$$2y^2 - 2 = 3x^2 - 5x + 2$$

$$2y^2 = 3x^2 - 5x + 4$$

Therefore,

$$y = \pm\frac{1}{\sqrt{2}}\sqrt{3x^2 - 5x + 4}$$

Clearly, LHS and RHS must agree at $(x, y) = (1,1)$, hence the sign of the RHS must be positive. Hence

$$\boxed{y = \frac{1}{\sqrt{2}}\sqrt{3x^2 - 5x + 4}}$$

</td><td>

1

1

</td><td>

The first mark was awarded for establishing the general solution.

There weren't many problems with this.

The second mark was obtained for giving the correct solution, given the boundary conditions. This meant solving explicitly for y and justifying, using boundary conditions, why the positive solution was obtained.

Many students left the expression in terms of y^2. Unfortunately, the problem requires that y itself be found explicitly.

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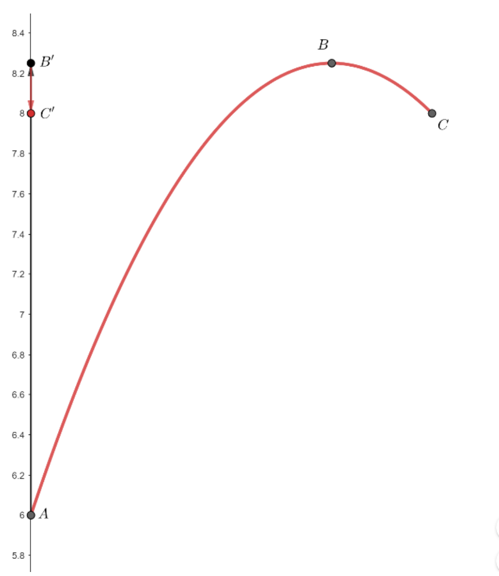
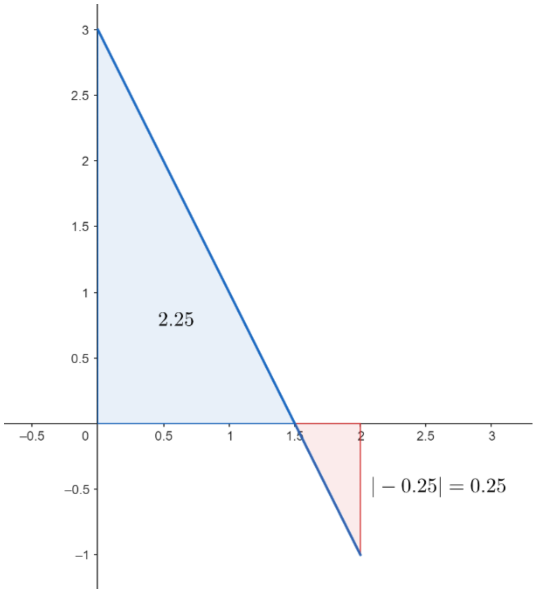
(b) The displacement equation: $x = -t^2 + 3t + 6$		
(i) Velocity: $\dot{x} = -2t + 3 \Longrightarrow \dot{x}(2) = -2(2) + 3 = -1\ \mathrm{us}^{-1}$ Acceleration: $\ddot{x} = -2 \Longrightarrow \ddot{x}(2) = -2\ \mathrm{us}^{-2}$	1	The first mark was awarded for both correct velocity and acceleration.
(i) Object may change direction about stationary points in its displacement, so we first look for t such that $\dot{x} = 0$: $-2t + 3 = 0 \rightarrow t = 3/2\ \mathrm{s}^{-1}$. Since $\ddot{x} = -2 < 0$ (i.e. second derivative is non-zero), the point $\left(2, x(2)\right)$ is a turning point. Hence the object will change direction after $t = 3/2$ seconds.	1	The second mark was awarded for correct determination of time at which velocity was zero. Generally, no problems.
(ii) The distance travelled in the first 2 seconds: the object has positive change in displacement from $t = 0$ to $t = 3/2$, then negative change in displacement from $t = 3/2$ to $t = 2$. The distance is the sum of the **magnitudes** of the differences in displacements:	1	First mark awarded for correct setup of the problem/identifying the need to determine distance, and not displacement. This also meant finding $x(1.5)$ and partitioning distances $(0 \leq t \leq 1.5, 1.5 \leq t \leq 2)$.
 The graph of the displacement equation (x vs t) is a quadratic. The 1D displacement is projected onto the vertical axis (x-axis). The total distance is the sum of the magnitudes, $AB' + B'C'$, **not** the displacement AC. $d = x\left(\frac{3}{2}\right) - x(0) + \left\lvert x(2) - x\left(\frac{3}{2}\right)\right\rvert$ $= -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 6 - [0 + 0 + 6] + \left\lvert -2^2 + 3(2) + 6 - \left(-\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 6\right)\right\rvert$ $= 2.25 + \lvert -0.25 \rvert$ $= 2.5$ units	1 1	The second mark was awarded for sufficient progress, relative to the cohort. The third mark was awarded for correct distance. The major flaw in responses here was the misunderstanding of distance vs displacement. Displacement concerns itself only with a net change in position, and the magnitude can underestimate the distance. Also, some students integrated the displacement equation: this won't work.

Alternatively, the distance is equivalent to the area bounded by the velocity curve, the t-axis and the lines $t = 0$ and $t = 2$:



$$\int_0^{\frac{3}{2}} -2t + 3 \ dt + \left| \int_{\frac{3}{2}}^{2} -2t + 3 \ dt \right| = \frac{5}{2} = 2.5 \text{ units}$$

The integral of **speed** (not velocity) is probably what candidates attempting this method intended.

You could see that integrating x vs t would not give distance, since the units of the ‘area’ found would have been [length] $\times$ [time].

Integrating speed against time gives units for the ‘area’ of $\left[\frac{\text{length}}{\text{time}}\right] \times [\text{time}] =$ [length], which is what you need.

Student Number Solutions

Question 9

a) | C | C | C | C | C | C | C | C | C |

10 spaces to divide up chocolates into 2 groups. ✓

8 spaces to divide up chocolates into 2 groups where both people have at least one chocolate. ✓

Both answers marked correct.

b)i) $\left.\begin{matrix} t=1 \\ M=3821 \end{matrix}\right\} \Rightarrow 3821 = 25 + 4875e^{-k}$

$e^{-k} = \frac{292}{375}$ ✓ sub in initial conditions and simplify

$-k = \ln \frac{292}{375}$

$k = 0.250172223\ldots$

$k = 0.25$ (2 dec. pl.)

ii) $\frac{dH}{dt} = 0.001H(250-H)$

$\frac{dt}{dH} = \frac{1}{0.001H(250-H)}$

$\frac{dt}{dH} = 4\left(\frac{1}{H} + \frac{1}{250-H}\right)$

$t = 4\ln|H| - 4\ln|250-H| + C$ ✓ antiderivative

$\left.\begin{matrix} t=0 \\ H=50 \end{matrix}\right\} \Rightarrow 0 = 4\ln 50 - 4\ln 200 + C$ ✓ find c

$C = 4\ln 4$

$t = 4\ln H - 4\ln(250-H) + 4\ln 4$ as $50 \le H \le 250$

Student Number Solutions

$t = 4\ln H - 4\ln(250-H) + 4\ln 4$

$-0.25t = \ln(250-H) - \ln H - \ln 4$

$-0.25t = \ln \frac{250-H}{4H}$

$e^{-0.25t} = \frac{250-H}{4H}$ ✓ simplify

$4e^{-0.25t} = \frac{250}{H} - 1$

$\frac{H}{250} = \frac{1}{1+4e^{-0.25t}}$

$H = \frac{250}{1+4e^{-0.25t}}$

iii) Need $H > M$

$\frac{250}{1+4e^{-0.25t}} > 25 + 4875e^{-0.25t}$

$250 > 25 + 4875e^{-0.25t} + 100e^{-0.25t} + 4 \cdot 4875(e^{-0.25t})^2$

$780(e^{-0.25t})^2 + 199e^{-0.25t} - 9 > 0$ ✓ forming quadratic

$e^{-0.25t} < \frac{-199-\sqrt{67681}}{1560}$ or $e^{-0.25t} > \frac{-199+\sqrt{67681}}{1560}$ $\Delta = 199^2 + 4 \cdot 780 \cdot 9 = 67681 > 0$

No real sol^{n}s $\quad t > -4\ln\left(\frac{\sqrt{67681}-199}{1560}\right) > 0$ ∴ t_1 t_2

no mark here $\rightarrow t > 12.95607\ldots$ years

$t > 12$ years 11.5 months after 1st March 2024

must answer question "in what year" for mark

∴ Population hawks will exceed population mice $\boxed{\text{in } 2037}$ ✓

MATHEMATICS Extension 1 : Question 10

Suggested Solutions	Marks	Marker's Comments
a) i) $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$		
let $t = \tan\theta$		
RHS $= \dfrac{1 - \frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$		
$= \dfrac{\frac{1+t^2-1+t^2}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}}$		
$= \dfrac{2t^2}{2}$	1	
$= t^2$		
$= \tan^2\theta$		
$=$ LHS		
ii) $\tan^2\frac{\pi}{12} = \dfrac{1-\cos\frac{\pi}{6}}{1+\cos\frac{\pi}{6}}$		
$= \dfrac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}$		
$= \dfrac{2-\sqrt{3}}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}}$	1	
$= \dfrac{(2-\sqrt{3})^2}{4-3}$		
$= (2-\sqrt{3})^2$		
$\therefore \tan\frac{\pi}{12} = 2-\sqrt{3}$, as $\tan\frac{\pi}{12} > 0$ since $\frac{\pi}{12}$ is the first quadrant.	1	must have some reason to why we take the positive solution.

MATHEMATICS Extension 1 : Question 10

Suggested Solutions	Marks	Marker's Comments
b) Method 1:		
number of ways = total − EE together		
$= \frac{9!}{2!2!} - \frac{8!}{2!}$	1	
$= 70560$	1	
Method 2:		
number of ways without Es × slot Es		
$= \frac{7!}{2!} \times \frac{8 \times 7}{2!}$	1	
$= 70560$	1	
(c) No. of triangles		
= total − vertical − horizontal − diagonal		
$= {}^{12}C_3 - 4 - 3 \times {}^4C_3 - 4$	1	
$= 200$	1	

Question 11

1.

a)

i.

$$\frac{d}{dx}(x\ln x - x) = \left(x \times \frac{1}{x} + 1 \times \ln x\right) - 1$$
$$= 1 + \ln x - 1$$
$$= \ln x \qquad \text{1 mark}$$

ii.

$$\int_{\sqrt{e}}^{e} \ln x\, dx = [x\ln x - x]_{\sqrt{e}}^{e} \qquad \text{1 mark for using part i correctly}$$
$$= [(e \times \ln(e)) - e] - \left[\sqrt{e}\ln\sqrt{e} - \sqrt{e}\right]$$
$$= [e - e] - \left[\sqrt{e}\ln\left(e^{\frac{1}{2}}\right) - \sqrt{e}\right]$$
$$= \sqrt{e} - \frac{\sqrt{e}}{2}\ln(e)$$
$$= \sqrt{e} - \frac{\sqrt{e}}{2}$$
$$= \frac{\sqrt{e}}{2} \qquad \text{1 mark for final answer}$$

b)

i.

Let $R\sin(x+\alpha) = \cos x - \sin x$

$\therefore R\sin x\cos\alpha + R\cos x\sin\alpha = \cos x - \sin x$

$R\cos\alpha = -1$ Equation 1

$R\sin\alpha = 1$ Equation 2 1 mark for getting both equations

(Equation 2) $\div$ (Equation 1) $\Rightarrow \tan\alpha = -1$

$\alpha = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

Using $\alpha = \frac{3\pi}{4}$ into Equation 2:

$R\sin\frac{3\pi}{4} = 1$

$\frac{R}{\sqrt{2}} = 1$

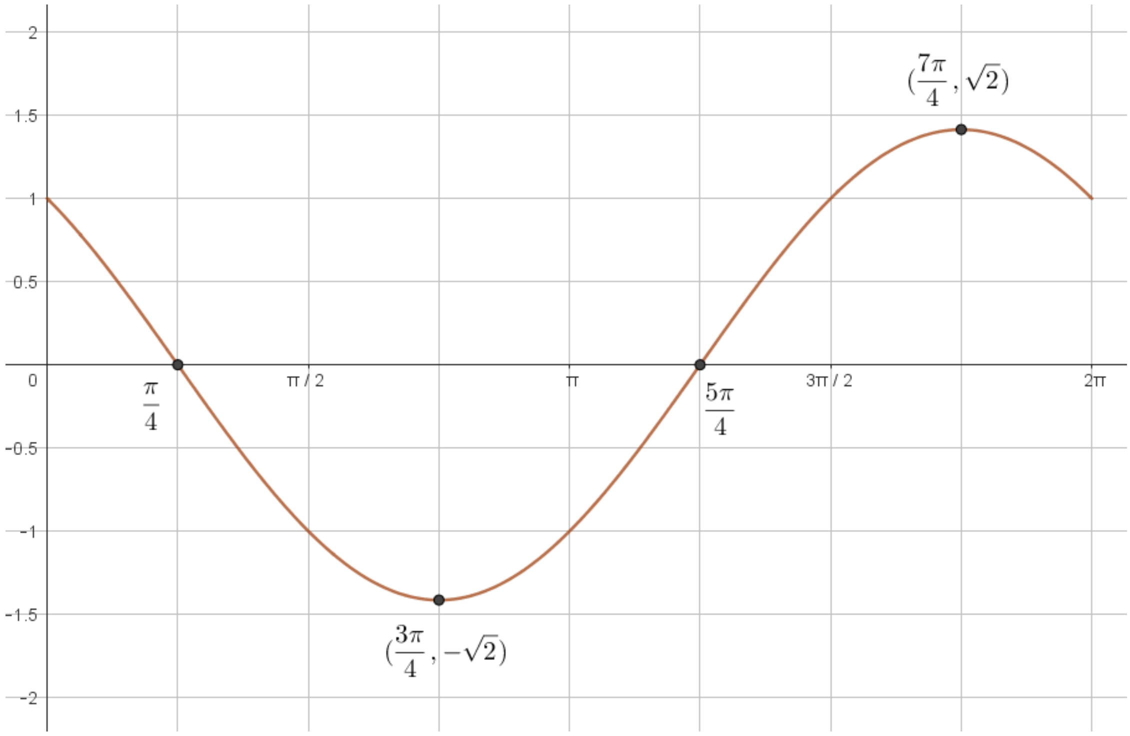
$R = \sqrt{2}$

$\cos x - \sin x = \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$ 1 mark for final answer

Other answers can include: $-\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$ or $-\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

ii.

Sketching $y = \cos x - \sin x \Rightarrow$ Sketching $y = \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$



1 mark for shape (1 cycle of a sinusoidal curve with the correct amplitude

1 mark for including all other important features

Student Number ______________

Question 12.

(a) (i) $\frac{k-4}{13-5} = \frac{k-4}{8} = \frac{-4}{5}$ $\therefore$ $k = 4 - \frac{32}{5} = \frac{-12}{5} = -2.4$ (1 mark)

(ii) $0 \le t \le 10$

$d = 0.5 \times 10 \times 4 = 20$

$0.5 \times 3 \times \frac{12}{5} + \frac{12}{5} \times x = 20$ $\therefore$ $x = 6\frac{5}{6}$

Time $= \frac{41}{6} + 13 = 19\frac{5}{6}$ seconds $= 19.83$ seconds.

(1 mark)

(b) $\cos 7x + \cos 5x = \cos 6x$ $\quad 0 \le x \le 2\pi$

$2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right) = \cos 6x$

$2\cos 6x \cos x = \cos 6x$

$\cos 6x(2\cos x - 1) = 0$ (1 mark)

$\cos 6x = 0$ or $\cos x = \frac{1}{2}$

For $\cos 6x = 0$ $\quad 0 \le x \le 2\pi$ and $0 \le 6x \le 12\pi$

$6x = (2n+1)\frac{\pi}{2}$ $\forall n \in \mathbb{Z}$ $\therefore$ $x = (2n+1)\frac{\pi}{12}$

$0 \le (2n+1)\frac{\pi}{2} \le 12\pi$

$0 \le 2n+1 \le 24$

$-1 \le 2n \le 23$

$-\frac{1}{2} \le n \le \frac{23}{2}$ $\therefore$ $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

$\therefore$ $x = (2n+1)\frac{\pi}{12}$, $n \in [0, 11]$

Also $\cos x = \frac{1}{2}$ $\therefore$ $x = \frac{\pi}{3}$ and $5\pi/3$

$\therefore$ $x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}, \frac{23\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{3}$

(2 marks)

- 2 marks for 14 correct solutions
- 1 mark for 6 or more correct solutions.

Student Number

(c) $2^{3^n}+1$ is divisible by 3^{n+1} $n \geq 0$.

Base Case: $n=0$

For $n=0$, $2^{3^0}+1 = 2^1+1 = 3$ and $3^{0+1} = 3^1 = 3$

$\therefore$ 3 is divisible by 3 $\therefore$ True for $n=0$.

Assume it is true for $n=k$

$2^{3^k}+1$ is divisible by 3^{k+1} $k \in \mathbb{Z}^+$ $(*)$ (1 mark)

For base case and the assumption

prove it is true for $n=k+1$

$2^{3^{k+1}}+1$ is divisible by 3^{k+2}

Firstly, you can write $(*)$ as

$$\exists m \in \mathbb{N}, \quad 2^{3^k}+1 = m3^{k+1} \qquad (**)$$

$$2^{3^k} = m3^{k+1}-1$$

$2^{3^{k+1}}+1$

$= 2^{3k \cdot 3}+1 = \left(2^{3k}\right)^3+1 = \left(m3^{k+1}-1\right)^3+1$ (1 mark)

$= m^3 3^{3k+3} - 3m^2 3^{2k+2} + 3m3^{k+1} - 1 + 1$

$= 3^{k+2}\left(m^3 3^{2k+1} - m^2 3^{k+1} + m\right)$

(1 mark).

$= N3^{k+2}$

where $N = m^3 3^{2k+1} - m^2 3^{k+1} + m \in \mathbb{N}$

$\therefore$ It is true for $n=k+1$

By mathematical induction, it is true for all $n \geq 0$.