KILLARA HIGH SCHOOL

TERM 2, 2000
YEAR 12

3/4 UNIT MATHEMATICS ASSESSMENT

Time Allowed: 2 hours

Attempt ALL questions.
Marks for each question are as indicated.
Approved calculators may be used.
Show ALL working clearly.
Marks may be deducted for careless or badly arranged work.
Begin each question on a new page.

Question 1 (12 marks)

(a) Evaluate $\int_0^2 \frac{dx}{4+x^2}$ **(3)**

(b) Differentiate $2x\sin^{-1}x$ **(3)**

(c) Evaluate $\int_2^{10} \frac{x}{\sqrt{x-1}}dx$ using the substitution $x = t^2 + 1$ **(3)**

(d) Find the coefficient of x^3 in the expansion of $(2x - \frac{1}{x})^{11}$ **(3)**

Question 2 (12 marks)

(a) Evaluate $\int_{-1}^{3}(\sin^{-1}x + \cos^{-1}x)dx$ **(2)**

(b) A circular plate of radius r is heated so that the area of the plate expands at a constant rate of $3\cdot 2cm^2\text{min}^{-1}$. At what rate does r increase when $r = 10cm$? **(3)**

(c) Prove the following identity:

$$\frac{\sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A - \sin A} = \tan 2A$$ **(3)**

(d) i) Sketch the graph of $y = \cos x$ for $0 \le x \le 2\pi$.

ii) By using (i) or otherwise, find those values of x satisfying $0 \le x \le 2\pi$ for which the geometrical series

$$1 + 2\cos x + 4\cos^2 x + 8\cos^3 x + \ldots\ldots\ldots$$

has a limiting sum. **(4)**

Question 3 (11 marks)

(a) Sketch showing all essential features

$$y = 2\cos^{-1} 3x$$ **(3)**

(b) Write down the inverse function of $y = e^x - 2$ in the form $y = g(x)$ and state the domain and range of $g(x)$. **(3)**

(c) The polynomial $P(x) = x^3 + bx^2 + cx + d$ has roots $0, 3$ and -3. **(5)**

i) Find b, c and d

ii) Without using calculus, sketch the graph of $y = P(x)$

(iii) Hence, or otherwise, solve the inequality $\dfrac{x^2 - 9}{x} > 0$

Question 4 (13 marks)

(a) The velocity, v metres per second, of a particle at time t seconds is given by

$$v = t^3 - 3t^2 + 4t - 6$$

Show that acceleration is never less than 1 metre per second per second. **(2)**

(b) Let $S_n = 1 \times 2 + 2 \times 3 + \ldots.. + (n-1) \times n$.

Use mathematical induction to prove that, for all integers n with $n \ge 2$,

$$S_n = \tfrac{1}{3}(n-1)n(n+1)$$ **(4)**

Question 4 continued

(c) In a flock of 1000 chickens, the number P infected with a disease at time t years is given by

$$P = \frac{1000}{1+ce^{-1000t}} \text{ where } c \text{ is a constant.}$$

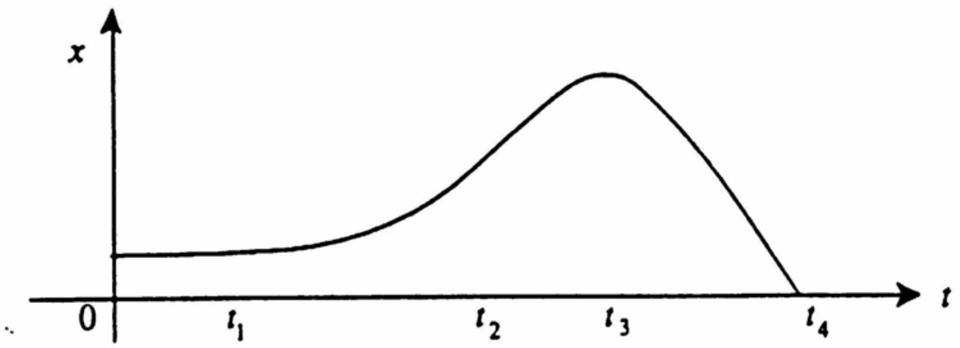
i) Show that, eventually, all the chickens will be infected.

ii) Suppose that when time $t = 0$, exactly one chicken was infected. After how many days will 500 chickens be infected?

iii) Show that $\frac{dP}{dt} = P(1000 - P)$. **(7)**

<u>*Question 5*</u> (12 marks)

(a)



A particle moves in a straight line and the above graph shows the distance x of the particle from a fixed point at time t.

i) Is the particle moving faster at time t_1 or at time t_2? Why?

ii) What is the velocity at time $t = 0$? Why?

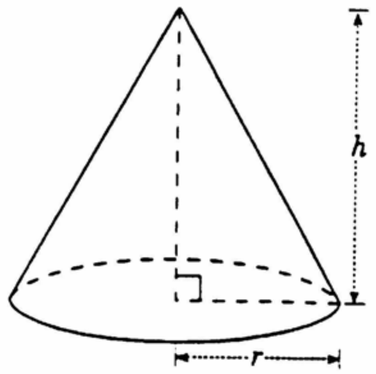
iii) Sketch the graph of the velocity v as a function of t. **(4)**

(b) Using the fact that $(1+x)^4(1+x)^9 = (1+x)^{13}$ show that

$${}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0 = {}^{13}C_4$$ **(3)**

Question 5 continued

(c)



Grain is poured at a constant rate of $0 \cdot 5$ cubic metres per second. It forms a conical pile, with the angle at the apex of the cone equal to 60°. The height of the pile is h metres, and the radius of the base is r metres.

i) Show that $r = \dfrac{h}{\sqrt{3}}$.

ii) Show that V, the volume of the pile, is given by $V = \dfrac{\pi h^3}{9}$.

(iii) Hence find the rate at which the height of the pile is increasing when the height of the pile is 3 metres. **(5)**

Question 6 (12 marks)

(a) i) On the same set of axes, sketch the graphs of $y = 2\sin\theta$ and $y = \theta$ for $-\pi \le \theta \le \pi$

ii) Use your sketch to find the number of solutions of the equation $2\sin\theta = \theta$ for $-\pi \le \theta \le \pi$ **(3)**

(b) The velocity $v$cm/s of a particle moving along the x axis at the end of t seconds is given by $v = \sqrt{9 - x^2}$, where x cm is the displacement from 0. Find an expression for x in terms of t, if initially the particle is 3cm to the left of 0. **(4)**

(c) Suppose $(7 + 3x)^{25} = \sum_{k=0}^{25} t_k x^k$

i) Use the binomial theorem to write an expression for $t_k . 0 \le k \le 25$.

ii) Show that $\dfrac{t_{k+1}}{t_k} = \dfrac{3(25-k)}{7(k+1)}$

Continued over

iii) Hence or otherwise find the largest coefficient t_k. You may leave your answer in the form $\binom{25}{k} 7^c 3^d$ **(5)**

Question 7 (12 marks)

(a) i) Prove that the graph of $y = \ln x$ is concave down for all $x > 0$.

ii) Sketch the graph of $y = \ln x$.

iii) Suppose $0 < a < b$ and consider the points $A(a, \ln a)$ and $B(b, \ln b)$ on the graph of $y = \ln x$.

Find the coordinates of the point P that divides the line segment AB in the ration 2 : 1.

iv) By using (ii) and (iii) deduce that $\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln(\frac{1}{3}a + \frac{2}{3}b)$. **(6)**

(b) i) Two cars, represented by points A and B, are travelling due east and due north respectively along two roads represented by two straight lines intersecting at O. At a certain instant, car A is 2 kilometres west of O and car B is 1 kilometre south of O, the former travelling at a constant speed of 1 kilometre per minute and the latter at constant speed V kilometres per minute.

(a) What value of V will cause a collision?

(b) Prove that, in the course of motion, the minimum distance between the cars is $\dfrac{|2V-1|}{\sqrt{(V^2+1)}}$ kilometres. **(6)**

Question 1 (12 marks)

a) $\frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_0^2 = \frac{1}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$ (1) (1)

$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi}{8}$ (1)

b) $2x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times 2$

$= 2\left[\frac{x}{\sqrt{1-x^2}} + \sin^{-1}x\right]$ (3)

c) $\int_2^{10} \frac{x}{\sqrt{x-1}}\,dx$

$x = t^2 + 1$
$dx = 2t\,dt$
$x = 10,\ t = 3$
$x = 2,\ t = 1$ (1)

$= \int_1^3 \frac{(t^2+1)2t\,dt}{\sqrt{t^2}}$ (1)

$= 2\int_1^3 (t^2+1)\,dt$

$= 2\left[\frac{t^3}{3} + t\right]_1^3$

$= 2\left[(9+3) - \left(\frac{1}{3}+1\right)\right]$

$= 21\frac{1}{3}$ (1)

d) $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

$= {}^{11}C_r\,(2x)^{11-r}(-x^{-1})^r$

$= A\,x^{11-r}x^{-r}$

$= A\,x^{11-2r}$ (1)

$\therefore 11 - 2r = 3$
$r = 4$ (1)

$\therefore T_5 = {}^{11}C_4\, 2^7(-1)^4 x^3$

$= 42240\,x^3$ (1)

Question 2 (12 marks)

a) $\int_{-1}^3 \frac{\pi}{2}\,dx = \frac{\pi x}{2}\Big]_{-1}^3$ (1)

$= 4 \times \frac{\pi}{2}$

$= 2\pi$ (1)

b) $\frac{dA}{dt} = 3\cdot2$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$
$= 20\pi$ when $r = 10$

$\frac{dA}{dr} = \frac{dA}{dt} \times \frac{dt}{dr}$

$20\pi = 3\cdot2\,\frac{dt}{dr}$

$\frac{dt}{dr} = \frac{20\pi}{3\cdot2}$

$\therefore \frac{dr}{dt} = \frac{3\cdot2}{20\pi} = \frac{0\cdot16}{\pi}$ (3)

$= 0\cdot051$ cm min^{-1}

c) LHS $= \frac{\sin A(\cos A - \sin A) + \sin A(\cos A + \sin A)}{\cos^2 A - \sin^2 A}$ (1)

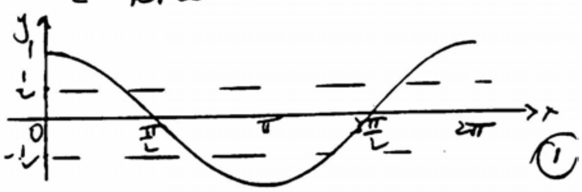
$= \frac{2\sin A\cos A}{\cos 2A}$ (1)

$= \frac{\sin 2A}{\cos 2A}$

$= \tan 2A$ (1)

$=$ RHS

d) i)



(1)

ii) lim sum if $-1 < r < 1$

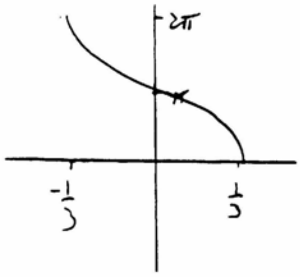
ie $-1 < 2\cos x < 1$ (1)

$-\frac{1}{2} < \cos x < \frac{1}{2}$

$\frac{\pi}{3} < x < \frac{2\pi}{3}$, $\frac{4\pi}{3} < x < \frac{5\pi}{3}$ (2)

Question 3 (11 marks)

a)



(1)

$-1 \leq 3x \leq 1$ (1)

$-\frac{1}{3} \leq x \leq \frac{1}{3}$

$0 \leq y \leq 2\pi$ (1)

b) $x = e^y - 2$

$e^y = x+2$

$y = \ln(x+2)$ (1)

D: $x > -2$ (1)

R: all real y (1)

c) $P(x) = x^3 + bx^2 + cx + d$

i) $x=0 \therefore 0 = d$

$x=3 \therefore 0 = 27 + 9b + 3c$

$-9 = 3b + c$ ①

$x=-3 \therefore 0 = -27 + 9b - 3c$

$9 = 3b - c$ ②

①+② $0 = 6b$

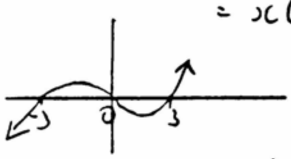
$b = 0$

$\therefore c = -9$ (2)

$b=0, \; c=-9 \; d=0$

ii) $P(x) = x^3 - 9x$

$= x(x+3)(x-3)$ (1)



iii) $\frac{x^2(x^2-9)}{x} > 0 \times x^2$

$x(x^2-9) > 0$

$\therefore -3 < x < 0 \; , \; x > 3$ (2)

Question 4 (13 marks)

a) $a = 3t^2 - 6t + 4$

min a when $t = \frac{--6}{6} = 1$

min $a = 3 - 6 + 4$

$= 1$ (2)

since min = 1 accel is never less than 1 m/s²

b) Prove $S_n = 1\times2 + 2\times3 + \dots + (n-1)n = \frac{1}{3}(n-1)n(n+$

b) Step 1 when $n=2$

LHS $= 1\times2$ $= 2$ RHS $= \frac{1}{3}\times1\times2\times3$ $= 2$

$\therefore$ true for $n=2$ 1

Step 2 Assume it is true for $n=k$

ie $S_k = 1\times2 + 2\times3 + \dots + (k-1)k = \frac{1}{3}(k-1)k(k+1)$

Step 3

Prove it is true for $n = k+1$

ie $S_{k+1} = 1\times2 + 2\times3 + \dots + (k-1)k + k(k+1)$

$= \frac{1}{3}[(k+1)-1][k+1][k+1+1]$ 2

$= \frac{1}{3}k(k+1)(k+2)$

Now $S_{k+1} = [1\times2 + 2\times3 + \dots + (k-1)k] + k(k+1)$

$= \frac{1}{3}(k-1)k(k+1) + k(k+1)$ by assumption

$= \frac{1}{3}k(k+1)[(k-1) + 3]$

$= \frac{1}{3}k(k+1)[k+2]$

$\therefore$ If it is true for $n=k$, it is true for $n=k+1$ 1

Since it is true for $n=1$, it is true for $n=1+1=$ and true for $n=3$ and so on for all $n \geq 2$ (4)

c) i) as $t \to \infty$, $\frac{c}{e^{1000t}} \to 1$

$\therefore$ $\to 1000$

$\therefore$ all chickens will be infected (1)

i) $t=0 \quad P=1$

$$1 = \frac{1000}{1+Ce^{0}} = \frac{1000}{1+C}$$

$$1+C = 1000$$

$$C = 999$$

$$P = \frac{1000}{1+999e^{-1000t}}$$

$$500 = \frac{1000}{1+999e^{-1000t}}$$

$$1+999e^{-1000t} = \frac{1000}{500} = 2$$

$$999e^{-1000t} = 1$$

$$e^{-1000t} = \frac{1}{999}$$

$$\ln\left(\frac{1}{999}\right) = -1000t$$

$$\frac{\ln \frac{1}{999}}{-1000} = t \qquad (3)$$

$$t = 0.0069 \text{ yrs}$$

$$\doteqdot 2.5 \text{ days}$$

iii) $P = 1000\left[1+Ce^{-1000t}\right]^{-1}$

$$\frac{dP}{dt} = 1000 \times -\left[1+Ce^{-1000t}\right]^{-2} \times \boxed{-1000C e^{-1000t}}$$

$$= \frac{1000^2}{\left[1+Ce^{-1000t}\right]^2} \times Ce^{-1000t}$$

$$= P^2 \times Ce^{-1000t}$$

$$= P^2 \times \left[\frac{1000}{P} - 1\right]$$

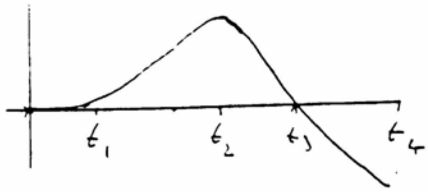
$$= P\left[1000 - P\right] \qquad (3)$$

Question 5 (12 marks)

a) i) t_2 since gradient is larger (1)

ii) 0 since gradient $= 0$ (1)

iii)



(2)

b) coefficient of x^4 in RHS is ${}^{13}C_4$

LHS: $\left({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4\right) \times$
$\left({}^9C_0 + {}^9C_1x + {}^9C_2x^2 + {}^9C_3x^3 + {}^9C_4x^4 + \dots + {}^9C_9x^9\right)$

terms in x^4 have coefficients

${}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0$

since coeffts of x^4 are the same for LHS and RHS then

${}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0 = {}^{13}C_4$ (1)

c) $\frac{dV}{dt} = 0.5 \text{ m}^3/\text{s}$

i) $\tan 30° = \frac{r}{h}$

$r = h\tan 30°$
$= h \times \frac{1}{\sqrt{3}}$
$= \frac{h}{\sqrt{3}}$ (1)

ii)

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h \quad \text{from i)}$$
$$= \frac{\pi h^3}{9} \qquad (1)$$

iii) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$0.5 = \frac{3\pi h^2}{9} \times \frac{dh}{dt}$$
$$= \frac{\pi \times 3^2}{3} \times \frac{dh}{dt} \quad \text{when } h=3$$

$$\frac{dh}{dt} = \frac{0.5}{3\pi} \qquad (3)$$
$$= 0.053 \text{ m/s}$$
$$= 53 \text{ cm/s}$$

Question 6 (12 marks)

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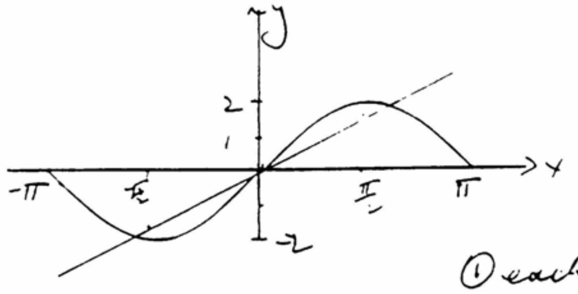


① each

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$
	-1.6	0	1.6

ii) 3 solutions ①

)
$$v=\sqrt{9-x^2}$$
$$\frac{dx}{dt}=\sqrt{9-x^2}$$
$$\frac{dt}{dx}=\frac{1}{\sqrt{9-x^2}} \quad ①$$
$$t=\sin^{-1}\frac{x}{3}+c \quad ①$$

$t=0$, $x=-3$
$$0=\sin^{-1}-1+c$$
$$=-\frac{\pi}{2}+c$$
$$c=+\frac{\pi}{2}$$
$$t=\sin^{-1}\frac{x}{3}+\frac{\pi}{2} \quad ①$$
$$\sin^{-1}\frac{x}{3}=t-\frac{\pi}{2}$$
$$\frac{x}{3}=\sin\left(t-\frac{\pi}{2}\right)$$
$$x=3\sin\left(t-\frac{\pi}{2}\right) \quad ①$$

c) $(7+3x)^{25}=\sum_{k=0}^{25} t_k x^k$

i) $T_{k+1}={}^{25}C_k\,7^{25-k}(3x)^k$
$$={}^{25}C_k\,7^{25-k}3^k x^k$$
$$\therefore t_k={}^{25}C_k\,7^{25-k}3^k \quad ①$$

ii) from i)
$$\frac{t_{k+1}}{t_k}=\frac{{}^{25}C_{k+1}\,7^{24-k}\,3^{k+1}}{{}^{25}C_k\,7^{25-k}\,3^k}$$
$$=\frac{3}{7}\times\frac{{}^{25}C_{k+1}}{{}^{25}C_k}$$
$$=\frac{3}{7}\times\frac{\frac{25!}{(k+1)!(25-k-1)!}}{\frac{25!}{k!(25-k)!}}$$
$$=\frac{3}{7}\times\frac{k!(25-k)!}{(k+1)!(24-k)!}$$
$$=\frac{3}{7}\times\frac{1}{(k+1)}\times(25-k)$$
$$=\frac{3(25-k)}{7(k+1)} \quad ②$$

iii) largest coefft when
$$\frac{t_{k+1}}{t_k}>1 \qquad \text{(assuming } t_{k+1}>t_k\text{.)}$$
$$\frac{3(25-k)}{7(k+1)}>1$$
$$75-3k>7k+7$$
$$68>10k$$
$$k<6.8$$
ie $k=6$

$\therefore$ largest coefft for T_{k+1} is $T_{6+1}=T_7$
$$={}^{25}C_7\,7^{18}3^7 \quad ②$$

Question 7 (12 marks)

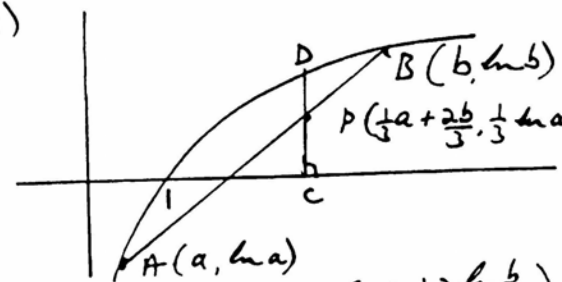
a) i) $\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$ ①

< 0 for $x > 0$

$\therefore$ concave down for $x > 0$

ii) ①



②

iii) $P = \left(\frac{a+2b}{3}, \frac{\ln a + 2\ln b}{3}\right)$ ②

iv) Draw CD through P ⊥ to x axis

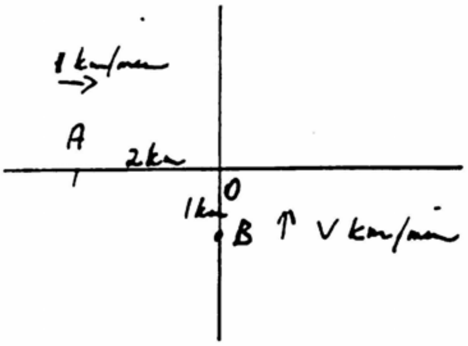
D has coordinates

$\left(\frac{1}{3}a + \frac{2b}{3}, \ln\left(\frac{1}{3}a + \frac{2b}{3}\right)\right)$ ②

$CP < CD$

$\therefore \frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln\left(\frac{1}{3}a + \frac{2b}{3}\right)$

b)



a) A will reach O in 2 min.

For B, $t = \frac{d}{V}$

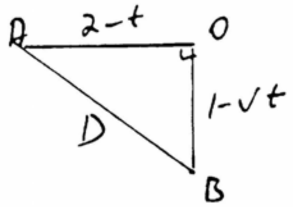
$= \frac{1}{V}$

$= 2$ for a collision.

$\therefore V = 0.5$

②

b)



①

At time t,

A is $2 - t$ km from O

B is $1 - Vt$ km from O

Let D be distance between the

$D^2 = (2-t)^2 + (1-Vt)^2$

$= 4 - 4t + t^2 + 1 - 2Vt + V^2t^2$

$= (1+V^2)t^2 - 2(2+V)t + 5$ eqn ①

②

min distance when $\frac{d(D^2)}{dt} = 0$

ie $2(1+V^2)t - 2(2+V) = 0$ eqn ②

check for min

$\frac{d^2(D^2)}{dt^2} = 2(1+V^2)$

> 0 $\therefore$ min

from ② $t = \frac{2(2+V)}{2(1+V^2)} = \frac{2+V}{1+V^2}$

Sub in ①

$D^2 = (1+V^2)\frac{(2+V)^2}{(1+V^2)^2} - 2(2+V)\frac{(2+V)}{(1+V^2)} + 5$ ①

$= \frac{(2+V)^2}{1+V^2} - \frac{2(2+V)^2}{1+V^2} + \frac{5(1+V^2)}{1+V^2}$

$= \frac{4V^2 - 4V + 1}{1+V^2}$

$= \frac{(2V-1)^2}{1+V^2}$

$\therefore D = \frac{|2V-1|}{\sqrt{V^2+1}}$ ①