KILLARA HIGH SCHOOL

YEAR 12, TERM 1 - ASSESSMENT 2001

MATHEMATICS EXTENSION 1

Time Allowed: 2 hours

Instructions:

All questions may be attempted.
All necessary working must be shown.
Approved calculators and templates may be used.
The possible marks for each question are shown in brackets adjacent to it.

Total Marks: 70

Question 1 **(10 marks)** **Marks**

(i) Differentiate with respect to x.

(a) $\log_e(\cos 3x)$ (b) $\sin(\log_e 3x)$ (c) $\tan^3 x$ **(6)**

(ii) Find the following indefinite integrals:

(a) $\int \frac{dx}{5x+3}$ (b) $\int \frac{x+1}{\sqrt{x}}$ **(4)**

Question 2 **(10 marks)**

(i) Evaluate $\lim_{x\to 0} \frac{\sin 3x}{2x}$ **(1)**

(ii) If $\log_5 \frac{1}{25} = \log_x 7$ evaluate $'x'$ in simplest surd form. **(3)**

(iii) Evaluate $\int \sin^5 x \cos^3 x dx$ using the substitution $u = \sin x$ **(4)**

(iv) Solve $\log_e[\log_e(\log_e x)] = 0$ **(2)**

Question 3 **(10 marks)**

(i) The curve $xy = 3$ is translated 3 units to the right and 1 unit up on the number plane. What is the equation of the resulting curve? **(2)**

(ii) Find all real numbers x, such that $|x + 2| \geq |x - 1|$ **(3)**

(iii) (a) Sketch the curve $y = 2\sin \pi x$ for $0 \leq x \leq 3$ showing its essential features. **(2)**

(b) Find the area bounded by the curve $y = 2\sin \pi x$, the X axis and the ordinates $x = 0$ and $x = 3$ leaving your answer in terms of π. **(3)**

Question 4 **(10 marks)**

(i) (a) Express $\sin 2x$ and $\tan x$ in terms of $\sin x$ and $\cos x$. **(1)**

(b) Hence, or otherwise, solve $\sin 2x = \tan x$, $0 \leq x \leq 2\pi$ **(3)**

(ii) (a) Show that $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$ **(2)**

(b) Hence evaluate the integral $\int_0^{\frac{\pi}{6}} \cos 5x . \cos 3x dx$, leaving your answer in simplest exact form. **(4)**

Question 5 **(10 marks)**

(i) (a) Sketch a parabola whose focus is (1, -1) and directrix is $x = -3$ **(1)**

(b) What is the equation of this parabola? **(2)**

(ii) (a) Use Simpson's Rule with 5 function values to find an approximation for

$$\int_0^8 2^x dx$$ **(3)**

(b) By integrating, find the value of $\int_0^8 2^x dx$, correct to the nearest integer. **(3)**

(c) Find the percentage error, when comparing Simpson's Rule with integration, correct to 1 decimal place. **(1)**

Question 6 **(10 marks)**

The two curves $y = e^x - 1$ and $y = 2e^{-x}$ intersect at point P.

(a) On the same diagram, sketch the curves $y = e^x - 1$ and $y = 2e^{-x}$ showing their essential features. **(2)**

(b) Show algebraically, that the coordinates of point P are $(\ln 2, 1)$ **(2)**

(c) For each of the equation $y = e^x - 1$ and $y = 2e^{-x}$, make x the subject of the equation. **(2)**

(d) Write an expression for the area bounded by the two curves and the Y axis. **(1)**

(e) Hence, or otherwise, find the size of the area bounded by the two curves $y = e^x - 1$ and $y = 2e^{-x}$ and the Y axis, leaving your answer in simplest exact form. **(3)**

Question 7 **(10 marks)**

(i) (a) Show that $\frac{d}{dx}(2x - \sin 2x) = 4\sin^2 x$ **(2)**

(b) Hence find the volume when the curve $y = \sin x$ is rotated about the X axis between the ordinates $x = \pi$ and $x = 2\pi$. **(3)**

(ii) A cylinder fits exactly inside a sphere of radius 1 metre. Find the maximum volume possible for the cylinder. **(5)**

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MATHS EXT 1 – SOLUTIONS

QUESTION 1 /10

i) a) $\frac{d}{dx}\log_e(\cos 3x)$

$= \frac{1}{\cos 3x} \cdot -3\sin 3x$ 2

$= \frac{-3\sin 3x}{\cos 3x}$ (or $-3\tan 3x$)

b) $\frac{d}{dx}\sin(\log_e 3x)$

$= \cos(\log_e 3x) \cdot \frac{1}{\cancel{3}x} \cdot \cancel{3}$

$= \frac{\cos(\log_e 3x)}{x}$ 2

c) $\frac{d}{dx}\tan^3 x = \frac{d}{dx}(\tan x)^3$

$= 3(\tan x)^2 \cdot \sec^2 x$

$= 3\tan^2 x(\sec^2 x)$ 2

ii) a) $\int \frac{dx}{5x+3} = \frac{1}{5}\ln(5x+3) + C$ 2

b) $\int \frac{x+1}{\sqrt{x}}\,dx$

$= \int x^{\frac{1}{2}} + x^{-1/2}\,dx$

$= \frac{2x^{3/2}}{3} + 2x^{1/2} + C$ 2

QUESTION 2 /10

i) $\lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\frac{3}{2}\times\frac{\sin 3x}{3x}$

$= \frac{3}{2}\times 1 = \frac{3}{2}$ 1

ii) $\log_5\frac{1}{25} = \log_x 7$

$\therefore \log_x 7 = -2$

$x^{-2} = 7$

$\frac{1}{x^2} = 7$

$\frac{1}{7} = x^2$

$\therefore x = +\frac{1}{\sqrt{7}}$ 3

(can't have negative base)

iii) $\int \sin^5 x\cos^3 x\,dx$.

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

Exp becomes

$\int u^5 \cdot \cos^2 x \cdot du$

$\int u^5 \cdot (1-u^2)\,du$

$= \int u^5 - u^7 \, du$

$= \frac{u^6}{6} - \frac{u^8}{8} + C$

$= \frac{\sin^6 x}{6} - \frac{\sin^8}{8} + C$ 4

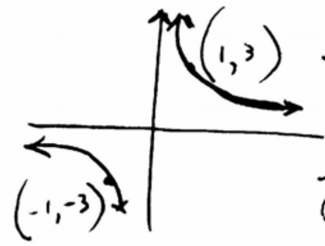
v) $\log_e[\log_e(\log_e x)] = 0$

$= \log_e(\log_e x) = 1$

$\log_e x = e$

$\therefore x = e^e$ 2

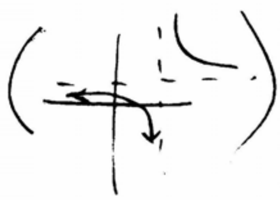
QUESTION 3 $\frac{}{10}$

i) 

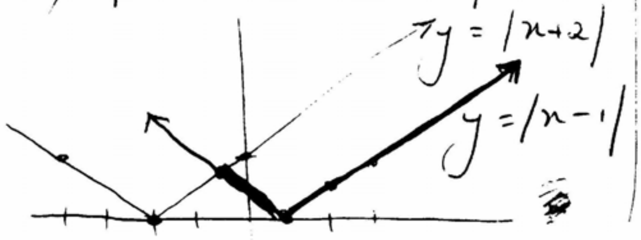
$\to 3 \uparrow 1$

$y = \frac{3}{x}$

$\therefore y = \frac{3}{x-3} + 1$



ii) $|x+2| \geq |x-1|$



$x+2 = -(x-1)$

$x+2 = -x+1$

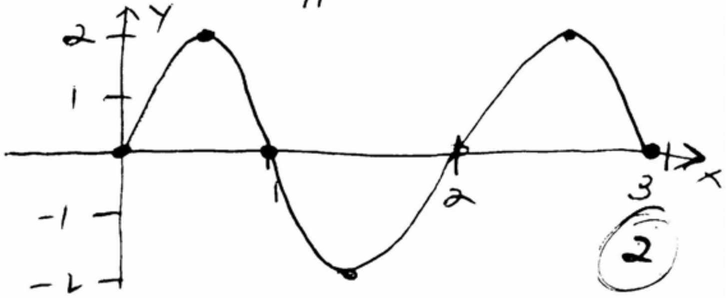
$2x = -1$

$x = -\frac{1}{2}$

$\therefore x \geq -\frac{1}{2}$ 3

iii) a) $y = 2\sin \pi x, \; 0 \leq x \leq 3$

period $= \frac{2\pi}{\pi} = 2$



2

b) $A = 3\int_0^1 2\sin \pi x \, dx$

$= 3 \times 2 \left[-\frac{\cos \pi x}{\pi} \right]_0^1$

$= \frac{-6}{\pi} \left[\cos \pi x \right]_0^1$

$= \frac{-6}{\pi} \left[\cos \pi - \cos 0 \right]$

$= \frac{-6}{\pi} \left[-1 - 1 \right]$

$= \frac{-6}{\pi} \times -2 = \frac{12}{\pi} u^2$

3

QUESTION 4 — /10

i) a) $\sin 2x = 2\sin x\cos x$

$\tan x = \dfrac{\sin x}{\cos x}$ (1)

b) $\sin 2x = \tan x$, $0 \le x \le 2\pi$

$$2\sin x\cos x = \frac{\sin x}{\cos x}$$

$$2\sin x(\cos x)^2 = \sin x$$

$$2\sin x(1-\sin^2 x) = \sin x$$

$$2\sin x - 2\sin^3 x = \sin x$$

$$2\sin^3 x - \sin x = 0$$

$$\sin x(2\sin^2 x - 1) = 0$$

$$\sin x = 0,\quad 2\sin^2 x = 1$$

$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \pm\frac{1}{\sqrt{2}}$$ (3)

$$\therefore x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$$

b) $$\int_0^{\pi/6} \cos 5x \cos 3x \, dx$$

$$= \int_0^{\pi/6} \frac{1}{2}(\cos 8x + \cos 2x)\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right]_0^{\pi/6}$$

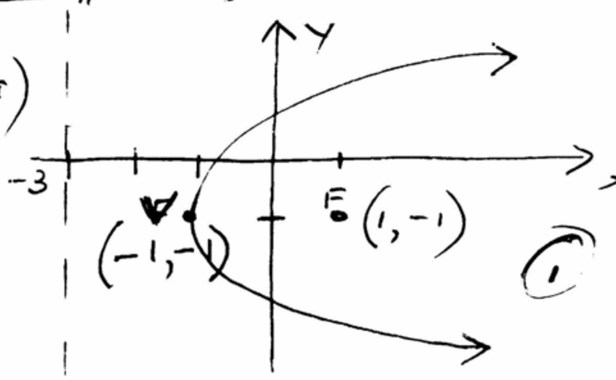
$$= \frac{1}{2}\left[\frac{\sin\frac{4\pi}{3}}{8} + \frac{\sin\frac{\pi}{3}}{2} - \left(\frac{\sin 0}{8} + \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-\sqrt{3}}{16} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{-\sqrt{3}}{32} + \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{32}$$ (4)

QUESTION 5 — /10

i) a)



(1)

b) $$(y+1)^2 = 4\times 2(x+1)$$

$$(y+1)^2 = 8(x+1)$$

$$y^2 + 2y + 1 = 8x + 8$$ (2)

$$y^2 + 2y - 8x - 7 = 0$$

ii) a) $$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$RHS = \frac{1}{2}\left[\cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B\right]$$

$$= \frac{1}{2}\left[2\cos A\cos B\right] = \cos A\cos B = LHS$$ (2)

iii) $$\int_0^8 2^x \, dx \doteq \frac{2}{3}\left[2^0 + 4\times 2^2 + 2\times 2^4 + 4\times 2^6 + 2^8\right]$$

$\doteqdot \frac{2}{3}\left[1+16+32+256+256\right]$

$= \frac{2}{3}\left[561\right]$

$= 374.$ (3)

b) $\int_0^8 2^x \, dx$

$= \left[\frac{1}{\ln 2} \cdot 2^x\right]_0^8$

$= \frac{1}{\ln 2} \cdot 2^8 - \frac{1}{\ln 2} \cdot 2^0$

$= \frac{1}{\ln 2}\left[256 - 1\right]$

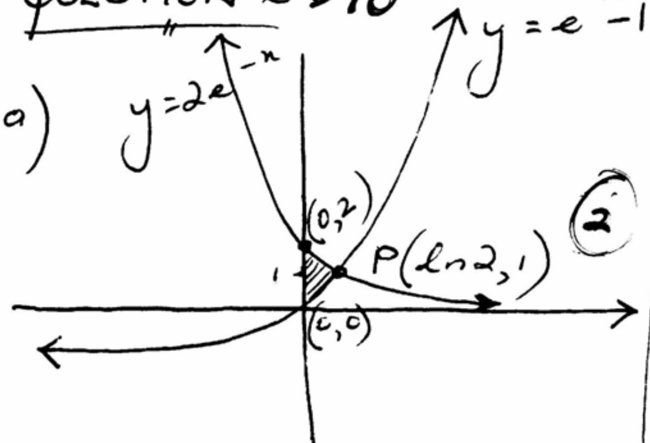
$= \frac{255}{\ln 2} \doteqdot 368$ (3)

c) $\frac{374 - 368}{368} \times 100\%$

$= 1.6\%$ (1 d.p) (1)

or 1.7% (depends on rounding off)

QUESTION 6 /10

a) 

(2)

b) $2e^{-x} = e^x - 1$

$\frac{2}{e^x} = e^x - 1$

$2 = e^{2x} - e^x$

$e^{2x} - e^x - 2 = 0$

$(e^x - 2)(e^x + 1) = 0$

$\therefore e^x = 2$ or -1 (no soln)

$\therefore e^x = 2$

so $x = \ln 2$

$y = e^{\ln 2} - 1$

$= 2 - 1 = 1$

$\therefore P = (\ln 2, 1)$

OR by substitution,

$y = e^x - 1$

$x = \ln 2,\ y = e^{\ln 2} - 1$

$= 2 - 1 = 1$

$\therefore (\ln 2, 1)$ satisfies (2)

$y = 2e^{-x},\ x = 2e^{-\ln 2}$

$= \frac{2}{e^{\ln 2}} = \frac{2}{2} = 1$

$\therefore (\ln 2, 1)$ satisfies

$\therefore P = (\ln 2, 1)$

c) $y = e^x - 1$

$y + 1 = e^x$

$x = \ln(y+1)$

$y = 2e^{-x}$

$\frac{y}{2} = \frac{1}{e^x}$

$e^x = \frac{2}{y}$

$x = \ln\left(\frac{2}{y}\right)$ (2)

d) $A = \int_1^2 \ln\left(\frac{2}{y}\right) dy$

$+ \int_0^1 \ln(y+1)\, dy$ (1)

OR, $\int_0^{\ln 2} 2e^{-x}\,dx - \int_0^{\ln 2} e^{x} - 1\,dx$

e) $A = \int_0^{\ln 2} 2e^{-x}\,dx - \int_0^{\ln 2} e^{x} - 1\,dx$

$= \left[\frac{2e^{-x}}{-1}\right]_0^{\ln 2} - \left[e^{x} - x\right]_0^{\ln 2}$

$= \left[\frac{-2}{e^{x}}\right]_0^{\ln 2} - \left[e^{x} - x\right]_0^{\ln 2}$

$= \left(\frac{-2}{e^{\ln 2}} - \frac{-2}{e^{0}}\right) - \left(e^{\ln 2} - \ln 2 - (e^{0} - 0)\right)$

$= \left(\frac{-2}{2} + 2\right) - \left(2 - \ln 2 - 1\right)$

$= -1 + 2 - 2 + \ln 2 + 1$

$= \ln 2.$

(3)

QUESTION 7 /10

i) a) $\frac{d}{dx}\left(2x - \sin 2x\right) = 4\sin^2 x$

LHS $= 2 - 2\cos 2x$

$= 2(1 - \cos 2x)$

$= 2\left(1 - (\cos^2 x - \sin^2 x)\right)$

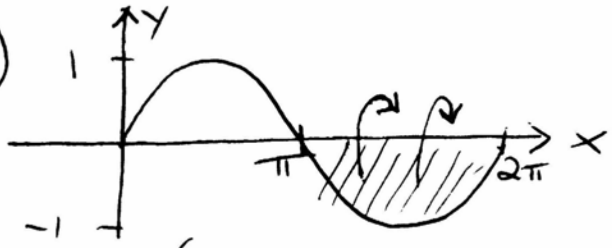
$= 2\left(1 - (1 - 2\sin^2 x)\right)$

$= 2\left(2\sin^2 x\right)$

$= 4\sin^2 x$

(2)

b)



$V = \pi \int y^2\,dx$

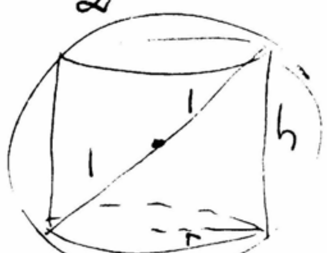
$= \pi \int \sin^2 x\,dx$

$= \frac{\pi}{4}\left[2x - \sin 2x\right]_0^{\pi}$

$= \frac{\pi}{4}\left[2\pi - \sin 2\pi - (0 - \sin 0)\right]$

$= \frac{\pi}{4}\left[2\pi\right] = \frac{\pi^2}{2}\,u^3$

ii) $V = \pi r^2 h$



$(2r)^2 + h^2 = 2^2$

$4r^2 + h^2 = 4$

$r^2 = \frac{4 - h^2}{4}$

$= 1 - \frac{h^2}{4}$

$V = \pi\left(1 - \frac{h^2}{4}\right)h$

$= \pi h - \frac{\pi h^3}{4}$

$V' = \pi - \frac{3\pi h^2}{4}$

st pts, $V' = 0$

$\pi - \frac{3\pi h^2}{4} = 0$

$4\pi - 3\pi h^2 = 0$

(5)

$4 = 3h^2$

$\frac{4}{3} = h^2$

$\therefore r^2 = 1 - \frac{\frac{4}{3}}{4}$

$= 1 - \frac{1}{3}$

$= \frac{2}{3}$

$\therefore V = \pi r^2 h$

$= \pi \cdot \frac{2}{3} \cdot \frac{2}{\sqrt{3}}$

$= \frac{4\pi}{3\sqrt{3}} \ u^3$

check max.

$V'' = -\frac{6\pi h}{4} < 0$

$\therefore$ max volume

is $\frac{4\pi}{3\sqrt{3}} \ u^3$.

or $\frac{4\pi}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}\pi}{9} \ u^3$