KILLARA HIGH SCHOOL
YEAR 12

EXTENSION MATEMATICS

APRIL 2002

Answer all questions and show all necessary working. Time allowed: 2 hours.

QUESTION 1 *(14 marks)*

a) Differentiate

(i) $y = e^x \cos x$ **1**

(ii) $y = e^{\sin x}$ **2**

(iii) $y = \dfrac{\log_e x}{x}$ **2**

b) Find the values of m for which the equation $y = e^m + 4e^{-m}$ has a stationary point and show this is a minimum turning point. **3**

c) Evaluate

(i) $\displaystyle\int_0^1 \frac{e^x}{1+e^x}dx$ **2**

(ii) $\displaystyle\int_0^1 \frac{\sin x}{3-\cos x}dx$ correct to 2 decimal places **2**

(iii) $\displaystyle\int_1^2 \frac{x^3+1}{x}dx$ exactly **2**

QUESTION 2 *(14 marks)*

a) (i) Show that $\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$ **3**

(ii) Hence or otherwise find the <u>exact</u> value of $\int_0^{\frac{\pi}{4}} \sin^4 x \ dx$ **2**

b) (i) Sketch the curve $y = 2 - \sin x$ for $0 \le x \le \pi$ **2**

(ii) Find the area between this curve and the X axis and ordinates $x = 0$ and $x = \pi$. **2**

(iii) If this area is rotated about the x axis, find the volume of the solid so formed (leaving your answer in terms of π). **2**

c) Show that the locus of the point of intersection of the tangent at the extremities of the focal chord of the parabola $x^2 = 4y$ is a straight line $y + 1 = 0$. **3**

QUESTION 3 *(13 marks)*

a) Given $f(x) = \sqrt{1 - x^2}$, evaluate $f'(\frac{3}{5})$ **2**

b) Solve $\log_3(9x - 2) = 2 + 2\log_3 x$ **2**

c) A particle moves along a staight line and its displacement x (metres) from a fixed point O is given by $x = 2\ln(t + 1)$

where t = time in seconds.

(i) What is the initial position of the particle? **1**

(ii) What is its initial velocity? **2**

(iii) What is its initial acceleration? **2**

(iv) When does the particle have a velocity of $1 \cdot 5 ms^{-1}$? **2**

(v) Show that the particle never comes to rest. **2**

QUESTION 4 *(12 marks)*

a) A southerly wind is blowing at a steady speed of 36 km/hr. Clouds are moving horizontally with the wind and heavy rain drops are falling to the ground 200m below.
(Assume no air resistance and take $g = 10ms^{-2}$)

(i) Find the time taken for a rain drop to reach the ground. **2**

(ii) Find (a) the speed, and **2**

(b) the angle **2**

at which the drop hits the horizontal ground.

(iii) At what angle does a drop hit the ground when the wind speed is doubled? **2**

b) A balloon is expanding so that its surface area is increasing at a rate of $0 \cdot 05cm^2/\text{sec}$.
Calculate the rate of change of volume when its diameter is 8cm. **4**

QUESTION 5 *(12 marks)*

(a) Sketch the graphs of

(i) $y = 2\tan^{-1}x$ **3**

and (ii) $y = \cos^{-1}\frac{x}{2}$ **3**

State their largest possible domain and range.

(b) What is the value of

(i) $\tan^{-1}(0)$ **1**

(ii) $\cos^{-1}(-\frac{1}{2})$ **1**

(c) What is the angle of inclination of the tangent to the curve $y = \sin^{-1}x$ at the origin? **2**

(d) What is the equation of the normal to the curve at this point? **2**

END OF PAPER

TERM 1 2002 SOLUTIONS

QUESTION 1 (14)

A. i $\frac{dy}{dx} = e^x(\cos x - \sin x)$ 1

ii $\frac{dy}{dx} = \cos x \cdot e^{\sin x}$ 2

iii $\frac{dy}{dx} = \frac{1 - \log_e x}{x^2}$ 2

B. $y = e^M + 4e^{-M}$

$\frac{dy}{dx} = e^M - 4e^{-M}$ 1

$= 0$ WHEN $e^M = \frac{4}{e^M}$

$M = \log_e 2$ 1

$\frac{d^2y}{dx^2} = e^M + 4e^{-M}$

> 0 AT $M = \log_e 2$ } 1

$\therefore$ MINIMUM

C. i $\int_0^1 \frac{e^x}{1+e^x}\,dx = \left[\log_e(1+e^x)\right]_0^1$ 1

$= \log_e(1+e) - \log_e 2$ 1

$\log_e\left(\frac{1+e}{2}\right)$ 1.17

ii $\int_0^1 \frac{\sin x}{3-\cos x}\,dx = \log_e(3-\cos x)\Big]_0^1$ 1

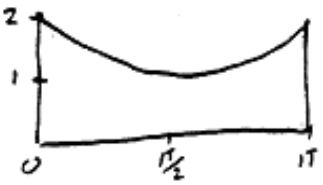
$= \left[\log_e(3-\cos x)\right]_0^1$ 1

$= \ln(3-\cos 1) - \ln 2$

iii $\int_1^2 \frac{x^3+1}{x} = \frac{x^3}{3} + \ln x\Big]_1^2$ 1

$= \frac{7}{3} + \ln 2$ 1

6 2B i



2

ii $A = \int_0^\pi 2 + 2\sin x\,dx$

$= 2x + \cos x\Big]_0^\pi$ 1

$= 2\pi - 2$ UNITS^2 1

iii $V = \pi\int_0^\pi y^2\,dx$

$= \pi\int_0^\pi (2-\sin x)^2\,dx$ 1

$= \pi\int_0^\pi 4 - 4\sin x + \sin^2 x$ $\frac{1}{2} - \frac{1}{2}\cos 2x$

$= \pi\left[\frac{9\pi}{2} + 4\cos x - \frac{1}{4}\sin 2x\right]_0^\pi$ 1

$= \pi\left[\frac{9\pi}{2} - 8\right]$

$= \frac{9\pi^2}{2} - 8\pi$ u^3 1

3 2C. $x^2 = 4y$ $A = 1$. LET P & Q BE PTS

EQUN TANGENT P: $y = Px - P^2$ 1

Q: $y = Qx - Q^2$

$(P-Q)x = (P^2 - Q^2)$

$x = P+Q$ 1

$y = PQ$ $(= -1)$

FOCAL CHORD $PQ = -1$. 1

$x = \frac{P^2-1}{P}$ $y = -1$

$\therefore y + 1 = 0$

QUESTION 2 (14)

A i $\sin^4 x = \sin^2 x \cdot \sin^2 x$

ii $= (1-\cos^2 x)(1-\cos^2 x)$

$= \frac{1}{4}(1-\cos 2x)^2$

$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}(1+\cos 4x)$

$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{8}$

$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

ii $\int_0^{\frac{\pi}{4}} \sin^4 x\,dx = \left[\frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]_0^{\frac{\pi}{4}}$ 1

$\frac{3\pi}{32}$

QUESTION 3

2 A. $f(x) = \sqrt{1-x^2}$ $f'(x) = \frac{-x}{\sqrt{1-x^2}}$

AT $x = \frac{3}{5}$

$f'(x) = -\frac{3}{4}$.

2 B. $\log_3(9x-2) = 2 + 2\log_3 x$

$\therefore \log_3 \frac{9x-2}{x^2} = 2$

$\frac{9x-2}{x^2} = 9$

$9x^2 - 9x + 2 = 0$

$x = \frac{1}{3}, \frac{2}{3}$

3 (CONT)

c. $x = 2\ln(t+1)$

$v = \frac{2}{t+1}$

$a = \frac{-2}{(t+1)^2}$

A. At $t=0$, $x=0$

B. $t=0$, $v = 2\ ms^{-1}$

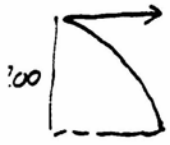
C. $t=0$, $a = -2\ ms^{-2}$

D) $1.5t = 2 - 1.5$

$t = \frac{1}{3}$ sec.

E) For rest, $\frac{2}{t+1} = 0$ — IMPOSSIBLE

Q. 4A)



$\dot{x} = 36 = 10\ ms^{-1}$

HORIZ

$\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = 10$ $\quad \dot{y} = -10t$

$x = 10t$ $\quad y = -5t^2 + 200$

A. AT GROUND $y = 0$

$\therefore -5t^2 + 200 = 0$

$t^2 = 40$

$t = 2\sqrt{10}$ secs.

B. AT GROUND $\dot{x} = 10\ ms^{-1}$

$\dot{y} = -20\sqrt{10}$

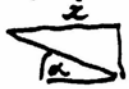
$V = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{10^2 + (20\sqrt{10})^2}$

$= \sqrt{4100}$

$\div 64.03\ ms^{-1}$

Let α = ANGLE



$\tan\alpha = \frac{20\sqrt{10}}{10} = 2\sqrt{10}$

$\alpha = 81°$ (APP)

C. If $\dot{x} = 20\ ms^{-1}$ $\tan\alpha = \frac{20\sqrt{10}}{20} = \sqrt{10}$

$\alpha = 72.27°$

4B) $\frac{dA}{dt} = 0.05$ $\quad A = 4\pi R^2$ ✓

$\frac{dA}{dR} = 8\pi R$

$\frac{dA}{dt} = \frac{dA}{dR} \times \frac{dR}{dt}$

$0.05 = 8\pi R \times \frac{dR}{dt}$

$\frac{dR}{dt} = \frac{0.05}{8\pi R}$ ✓

Now $\frac{dV}{dt} = \frac{dV}{dR} \times \frac{dR}{dt}$ ✓

$= 4\pi R^2 \times \frac{0.05}{8\pi R}$

$= \frac{R}{40}$

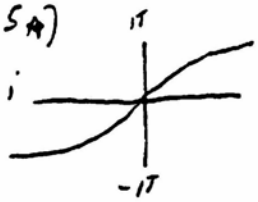
WHEN $D = 8$, $R = 4$,

$\frac{dV}{dt} = \frac{4}{40} = \frac{1}{10}$ ✓

$\therefore$ VOL INCR. AT $0.1\ cm^3 s^{-1}$

(Q. 5A)

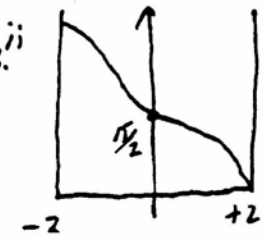
A) i



DOMAIN: $-\infty < x < \infty$

RANGE: $-\pi < y < \pi$.

B) ii



DOMAIN: $-2 \le x \le 2$

RANGE $0 \le y \le \pi$.

B) i $\tan^{-1}(0) = 0$

ii $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$.

C.; $y = \sin^{-1}x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ ✓

At $x = 0$, $\frac{dy}{dx} = 1$ ✓

$\therefore$ ANGLE $= \frac{\pi}{4}$ ✓

D. GRAD OF NORMAL $= -1$.

$y = -x$ EQN.

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