KILLARA HIGH SCHOOL

YEAR 12 EXTENSION 1 MATHEMATICS

TERM 1, 2003

Time Allowed: 2 hours

Instructions: Begin each section on a new page.
Write solutions on one side of the paper only.
Answers without appropriate working may not be awarded marks.
Marks may not be awarded for careless or badly arranged work.
Board-approved calculators may be used.

Total: 95 marks.

SECTION A: TRIGONOMETRIC FUNCTIONS (25 marks)

Marks

1. Evaluate $\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$ **1**

2. Find the general solution for all angles θ for which $\cos 2\theta = \cos\theta$ **4**

3. i) Differentiate $y = \frac{\sin x}{x}$ **1**

   ii) Find $\frac{d}{dx}(2x - \sin 2x)$ as a simple function of $\sin x$. **3**

4. i) Find $\int \tan^2 x \, dx$ **2**

ii) Evaluate $\int_{\frac{\pi}{4}}^{\pi} \sin x \cos x \, dx$ **4**

iii) Use the substitution $x = 4\tan\theta$ to find $\int \frac{dx}{(x^2+16)^{\frac{3}{2}}}$ **5**

5. Determine the exact volume of the solid formed when the curve $y = 3 + 2\sin x$, between $x = 0$ and $x = \pi$, is rotated about the x – axis. **5**

SECTION B: EXPONENTIAL AND LOGARITHMIC FUNCTIONS (26 marks)

Start a new page.

Marks

1. i) Given $e^{2\cdot0974} = x$ (correct to 5 significant figures), work out the natural logarithm of $\frac{1}{x}$. **2**

ii) Given $\log_a 3 = 1\cdot0986$, what is $\log_a \sqrt{3}$? **2**

2. i) If $y = e^{3x}$, show that $\frac{d^2y}{dx^2} = 9y$ **3**

ii) If $y = \frac{3}{\log_e x}$, find the exact value of $f'(e)$. **4**

iii) Given that $y = e^x\sqrt{1-x}$, show that $\frac{dy}{dx} = \frac{e^x(1-2x)}{2\sqrt{1-x}}$. **4**

3. Show $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$. **6**

Hence evaluate $\displaystyle\int_1^2 \frac{x^2}{x+1}\,dx$. (Leave answer in exact form.)

4. Evaluate $\displaystyle\int_0^2 \frac{3e^{2x}}{1+e^{2x}}\,dx$, correct to 2 decimal places. **5**

<u>SECTION C: INVERSE FUNCTIONS.</u> (24 marks)

Start a new page.

Marks

1. If $y = \dfrac{3x+5}{x-4}$, express the inverse function as a function of x. **4**

2. i) Sketch the graph of the function $y = \sin^{-1}\left(\dfrac{x}{2}\right)$ **1**

ii) State the domain and range of the function. **2**

iii) Find the exact equation of the tangent to the curve $y = \sin^{-1}\left(\dfrac{x}{2}\right)$ **4**

at the point where $x = 1$.

3. Find the exact value of: **4**

i) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

ii) $\sin^{-1}\left(-\dfrac{1}{2}\right)$

iii) $\tan\left(\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right)$

4. Given $\int_0^1 \frac{1}{x^2+3}\,dx = k\pi$, find the value of the constant k. **4**

5. i) Show $\frac{d}{dx}\ln(\cos x) = -\tan x$ **1**

ii) Sketch the curve $y = \tan^{-1} x$, then find the exact area between this curve, the x-axis and the line $x = 1$. **4**

SECTION D: APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD. (20 Marks)

Start a new page.

Marks

1. The carbon isotope C14 decays at a rate proportional to its mass. Archaeological testing suggests 50% decay takes 5 580 years. A fossil contains 40% of the amount of C14 that it had initially. Estimate the age of the fossil to the nearest year. **7**

2. A spherical balloon is being inflated and its volume increases at the constant rate of 50 mm^3 per second. At what rate is its surface area increasing when the radius is 20 mm.? **5**

(Note: $V = \frac{4}{3}\pi r^3$ $\quad\quad$ $S.A. = 4\pi r^2$)

3. The velocity $v\ ms^{-1}$ of a particle moving in a straight line is given by $v = \frac{1}{1+t}$.

i) The particle is initially at the origin O. Show that $x = \ln(1+t)$. **3**

ii) Sketch the function $x = \ln(1+t)$, and hence explain that the particle is moving away from O as t increases. **2**

iii) Find the acceleration when $t = 0$. **3**

End of paper.

K.H.S. YEAR 12. EXT. I MATHEMATICS.

TERM 1, 2003 SOLUTIONS.

SECTION A: TRIG. FNS.

① i) $\lim_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2.$ (1.

② $\cos 2\theta = \cos\theta.$

$2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0.$ ✓

$(2\cos\theta + 1)(\cos\theta - 1) = 0.$

$\therefore \cos\theta = -\frac{1}{2}$ or $\cos\theta = 1.$ ✓

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$, $\theta = 0$

In general:

$\begin{cases} \theta = 2n\pi \pm \frac{2\pi}{3} & ✓ \\ \theta = 2n\pi. & ✓ \end{cases}$ (4

③ i) $y = \frac{\sin x}{x.}$ $\left(y = \frac{u}{v}\right).$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2.}$

$= \frac{x\cos x - \sin x}{x^2.}$ (1

ii) $y = 2x - \sin 2x$

$\frac{dy}{dx} = 2 - 2\cos 2x$ ✓

$= 2(1 - \cos 2x)$

$= 2\{1 - (1 - 2\sin^2 x)\}$ ✓

$= 2(2\sin^2 x)$

$= 4\sin^2 x.$ ✓ (3

④ i) $\int \tan^2 x \, dx.$

$= \int (\sec^2 x - 1)\, dx$ ✓

$= \tan x - x + c.$ ✓ (2.

ii) $\int_{\frac{\pi}{4}}^{\pi} \sin x \cos x \, dx.$

$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi} 2\sin x\cos x\, dx.$ ✓

$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi} \sin 2x \, dx.$ ✓

$= \frac{1}{2}.\left[\frac{1}{2}x - \cos 2x\right]_{\frac{\pi}{4}.}^{\pi}$ ✓

$= \frac{1}{4}(-1 - 0)$

$= -\frac{1}{4}.$ ✓ (4

iii) $\displaystyle\int \frac{dx}{(x^2+16)^{3/2}}$

$$x = 4\tan\theta$$
$$\frac{dx}{d\theta} = 4\sec^2\theta.$$
$$dx = 4\sec^2\theta\, d\theta.$$

$$= \int \frac{4\sec^2\theta\, d\theta}{\{(4\tan\theta)^2+16\}^{3/2}}$$

$$= \int \frac{4\sec^2\theta\, d\theta}{(16\tan^2\theta+16)^{\frac{3}{2}}} \quad ✓$$

$$= \int \frac{4\sec^2\theta\, d\theta}{\{16(\tan^2\theta+1)\}^{3/2}}$$

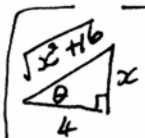
$$= \int \frac{4\sec^2\theta\, d\theta}{(16\sec^2\theta)^{3/2}} \quad ✓$$

$$= \int \frac{4\sec^2\theta\, d\theta}{64\sec^3\theta}$$

$$= \frac{1}{16}\int \frac{1}{\sec\theta}\, d\theta \quad ✓$$

$$= \frac{1}{16}\int \cos\theta\, d\theta$$

$$= \frac{1}{16}\sin\theta + C \quad ✓$$



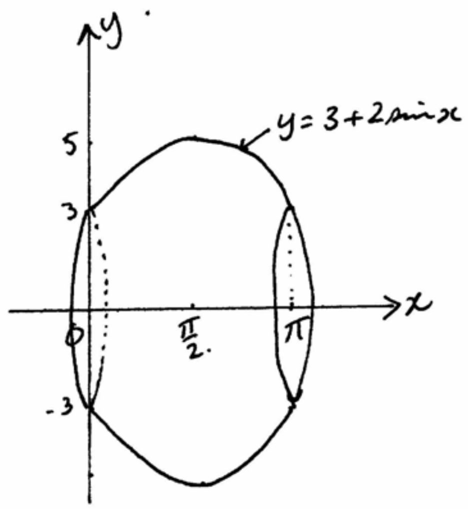
$$\sin\theta = \frac{x}{\sqrt{x^2+16}}$$

$$= \frac{x}{16\sqrt{x^2+16}} + C \quad ✓ \quad (5)$$

$$\frac{x}{16(x^2+16)^{\frac{1}{2}}} + C$$

PS. (2).

(5)



$$y = 3 + 2\sin x.$$

$$V = \pi\int y^2\, dx.$$

$$= \pi\int_0^{\pi} (3+2\sin x)^2\, dx \quad ✓$$

$$= \pi\int_0^{\pi} (9+12\sin x+4\sin^2 x)\, dx. \quad ✓$$

$$= \pi\int_0^{\pi} [9+12\sin x+2(1-\cos 2x)]\, dx \quad ✓$$

$$= \pi\int_0^{\pi} (9+12\sin x+2-2\cos 2x)\, dx$$

$$= \pi\int_0^{\pi} (11+12\sin x-2\cos 2x)\, dx$$

$$= \pi\Big[11x - 12\cos x - \frac{\cancel{2}}{\cancel{2}}\sin 2x\Big]_0^{\pi} \quad ✓$$

$$= \pi\Big[(11\pi - 12(-1) - 0) - (0 - 12(1) - 0)\Big]$$

$$= \pi.\,[11\pi + 24]$$

$$= \underline{11\pi^2 + 24\pi \quad u^3.} \quad ✓$$

(5.)

Total 25.

SECTION B: EXPONENTIAL & LOG. FNS.

① i) $e^{2.0974} = x$

$\log_e x = 2.0974$ ✓

$\log_e\left(\frac{1}{x}\right) = \log_e 1 - \log_e x$

$= 0 - 2.0974$

$= -2.0974$ ✓ (2

ii) $\log_a 3 = 1.0986$

$\log_a \sqrt{3} = \log_a 3^{\frac{1}{2}}$

$= \frac{1}{2}\log_a 3$ ✓

$= \frac{1}{2}(1.0986)$

$= 0.5493$. ✓ (2.

② i) $y = e^{3x}$.

$\frac{dy}{dx} = 3e^{3x}$. ✓

$\frac{d^2y}{dx^2} = 9e^{3x}$ ✓

$= 9y$. ✓ (3.

ii) $y = \frac{3}{\log_e x}$

$= 3(\log_e x)^{-1}$.

$\frac{dy}{dx} = -3.(\log_e x)^{-2} \times \frac{1}{x}$. ✓

ie. $f'(x) = \frac{-3}{x(\ln x)^2}$. ✓

$f'(e) = \frac{-3}{e(\log_e e)^2}$ ✓

$= \frac{-3}{e}$. ✓ (4

iii) $y = e^x\sqrt{1-x}$.

$u = e^x \quad v = (1-x)^{\frac{1}{2}}$

$u' = e^x \quad v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}.(-$

$\frac{dy}{dx} = uv' + vu'$.

$= e^x \times \frac{1}{2} \times \frac{-1}{\sqrt{1-x}}$ ✓ $+ \sqrt{1-x} \times e^x$ ✓

$= e^x\left(\frac{-1}{2\sqrt{1-x}} + \sqrt{1-x}\right)$ ✓

$= e^x\left(\frac{-1+2(1-x)}{2\sqrt{1-x}}\right)$

$= \frac{e^x(1-2x)}{2\sqrt{1-x}}$. ✓ (4

③ Show $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$.

R.H.S. $x - 1 + \frac{1}{x+1}$

$= \frac{(x-1)(x+1)+1}{x+1}$ ✓

$= \frac{x^2-1+1}{x+1}$ ✓

$= \frac{x^2}{x+1}$ ✓

$= L.H.S.$

$\therefore \int_1^2 \frac{x^2}{x+1}\,dx = \int_1^2\left(x - 1 + \frac{1}{x+1}\right)dx$

$= \left[\frac{x^2}{2} - x + \ln(x+1)\right]_1^2$ ✓

$= \left(\frac{4}{2} - 2 + \ln 3\right) - \left(\frac{1}{2} - 1 + \ln 2\right)$ ✓

$= \frac{1}{2} + \ln 3 - \ln 2$.

$= \frac{1}{2} + \ln\left(\frac{3}{2}\right)$. ✓ (6

(4). $\displaystyle\int_0^2 \frac{3e^{2x}}{1+e^{2x}}\,dx.$

$$= \frac{3}{2}\int_0^2 \frac{2e^{2x}}{1+e^{2x}}\,dx. \quad ✓$$

$$= \frac{3}{2}\left[\ln(1+e^{2x})\right]_0^2. \quad ✓$$

$$= \frac{3}{2}.\left\{(\ln(1+e^4)) - \ln(1+e^0)\right\}$$

$$= \frac{3}{2}\left(\ln(1+e^4) - \ln 2\right) ✓$$

$$= \frac{3}{2}\ln\left(\frac{1+e^4}{2}\right) \quad ✓$$

$$\doteqdot 4\cdot 99. \ (2\ d.p.)\ . ✓ \quad (5.$$

Total 26.

SECTION C: INVERSE FNS.

① $y = \dfrac{3x+5}{x-4}$.

f^{-1}: $x = \dfrac{3y+5}{y-4}$. ✓

$x(y-4) = 3y+5$

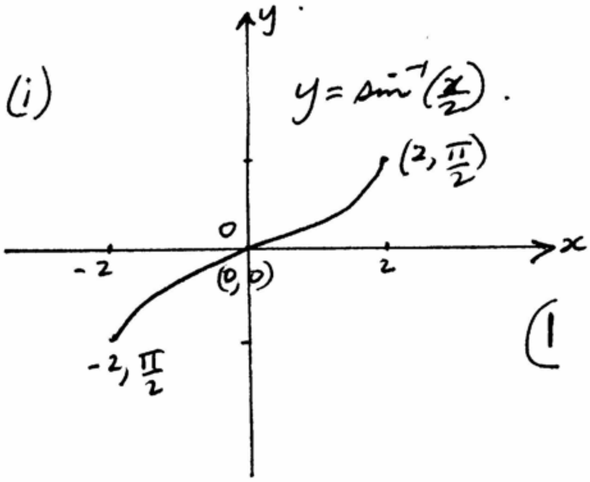
$xy - 4x = 3y+5$. ✓

$xy - 3y = 4x+5$

$y(x-3) = 4x+5$ ✓

$y = \dfrac{4x+5}{x-3}$. ✓ (4

② (i)



(1

ii) Dom: $-2 \leq x \leq 2$ ✓

Range: $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$. ✓ (2

iii). $y = \sin^{-1}\left(\dfrac{x}{2}\right)$.

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-x^2}}$. ✓

When $x = 1$, $y = \dfrac{\pi}{6}$ $\left(1, \dfrac{\pi}{6}\right)$.

$x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-1}}$ ✓

ie. $m = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$.

Eqn. tangent:

$y - y_1 = m(x - x_1)$.

$y - \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{3}(x-1)$ ✓

$6y - \pi = 2\sqrt{3}(x-1)$

$= 2\sqrt{3}x - 2\sqrt{3}$ ✓ (4

i.e. $2\sqrt{3}x - 6y + \pi - 2\sqrt{3} = 0$.

③ i) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}$. ✓

ii) $\sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$. ✓

iii) $\tan\left(\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right)$

$= \tan \dfrac{\pi}{6}$. ✓

$= \dfrac{1}{\sqrt{3}}$. ✓ (4

④ $\displaystyle\int_0^1 \frac{1}{x^2+3}\, dx = k\pi$.

L.H.S. $\displaystyle\int_0^1 \frac{1}{x^2+3}\, dx$

$= \displaystyle\int_0^1 \frac{1}{x^2+(\sqrt{3})^2}\, dx$.

$= \left[\dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right)\right]_0^1$ ✓

$= \dfrac{1}{\sqrt{3}}\left(\dfrac{\pi}{6} - 0\right)$ ✓

$= \dfrac{\pi}{6\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ ✓

$= \dfrac{\sqrt{3}}{18}\pi$. (4

$\therefore k = \dfrac{\sqrt{3}}{18}$. $\left(\text{or } \dfrac{1}{6\sqrt{3}}\right)$ ✓

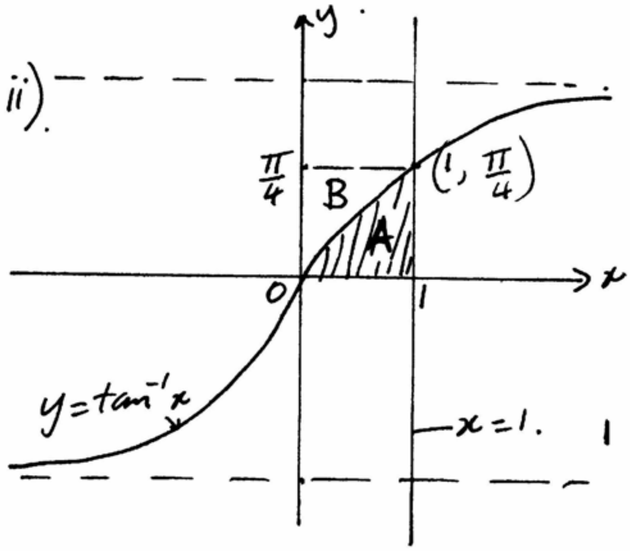
⑤ i) $y = \ln(\cos x)$

$$\frac{dy}{dx} = -\frac{\sin x}{\cos x}$$

$$= -\tan x.$$ (1.

i.e. $\frac{d}{dx}\ln(\cos x) = -\tan x.$

ii)



1

$y = \tan^{-1} x$

$\therefore x = \tan y.$

Area A $= \text{Rect.}\left(\frac{\pi}{4}\times 1\right) - $ Area B.

Area B $= \int_0^{\frac{\pi}{4}} x\, dy.$

$$= \int_0^{\frac{\pi}{4}} \tan y\, dy. \quad ✓$$

$$= -\Big[\ln(\cos x)\Big]_0^{\frac{\pi}{4}}.$$

$$= -\left\{\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right\}$$

$$= -\left\{\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right\}.$$

$$= -\left(\ln\left(\frac{1}{\sqrt{2}}\right) - 0\right) \quad ✓$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) \quad (= \ln\sqrt{2})$$

p.s. (6.)

$\therefore$ Area A $= \frac{\pi}{4} - -\ln\left(\frac{1}{\sqrt{2}}\right)$

$$= \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right).$$

$$\left\{\begin{aligned} &= \frac{\pi}{4} + \ln 1 - \ln\sqrt{2} \quad ✓ \\ &= \frac{\pi}{4} - \ln\sqrt{2}. \end{aligned}\right.$$ (4

Total 24.

SECTION D: APPLICATIONS OF CALCULUS TO PHYSICAL WORLD.

① . $M = M_0 e^{-kt}$.

$\frac{1}{2} \cancel{M_0} = \cancel{M_0} e^{-5580k}$. ✓

$-5580k = \log_e\left(\frac{1}{2}\right)$ ✓

$= \ln 1 - \ln 2$

$= -\ln 2$.

$\therefore 5580k = \ln 2$.

$k = \frac{\ln 2}{5580}$. ✓

When $M = 0{\cdot}4 M_0$.

$0{\cdot}4 \cancel{M_0} = \cancel{M_0} e^{-kt}$. ✓

$-kt = \ln 0{\cdot}4$

$t = \frac{\ln 0{\cdot}4}{-k}$. ✓

$= \ln 0{\cdot}4 \times \frac{-5580}{\ln 2}$ ✓

$= 7376$.

i.e. it reduces to 40% in ✓ 7376 years.

(7

② . $V = \frac{4}{3}\pi r^3$ ✓

$\frac{dV}{dr} = 4\pi r^2$ ✓ given $\frac{dV}{dt} = 50$.

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$. ✓

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$

$= 8\pi r \times \frac{1}{4\pi r^2} \times 50$ ✓

$= \frac{100}{r}$.

When $r = 20$

$\frac{dA}{dt} = \frac{100}{20}$

$= 5 \text{ mm}^2 \text{ s}^{-1}$. ✓ (5.

③ i) $v = \frac{1}{1+t}$.

$x = \int \frac{1}{1+t}\, dt$

$= \ln(1+t) + c$. ✓

When $x = 0$, $t = 0$.

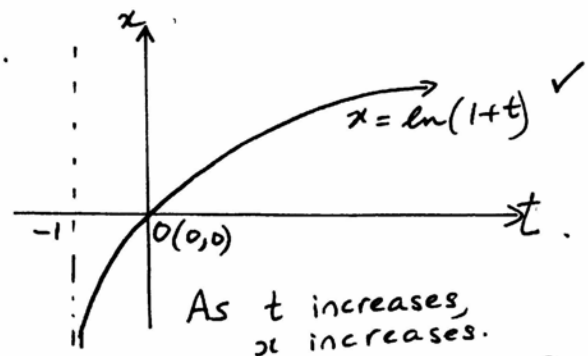
$0 = \ln(1) + c$. ✓

$= 0 + c$

$\therefore c = 0$.

$\therefore x = \ln(1+t)$ ✓ (3

ii).



✓

As t increases, x increases.

$\therefore$ moves away from O. ✓

(2.

iii) $v = \frac{1}{(1+t)}$

$= (1+t)^{-1}$.

$a = \frac{d}{dt}(1+t)^{-1}$. ✓

$= -1(1+t)^{-2}(1)$

$= \frac{-1}{(1+t)^2}$. ✓

When $t = 0$,

$a = \frac{-1}{1}$

$= -1 \text{ m s}^{-2}$. ✓ (3.

Total 20.