+solutions

KILLARA HIGH SCHOOL

YEAR 11 TERM 4 EXAMINATION
2004

EXTENSION MATHEMATICS

Time allowed: 1 hour
Total marks: 45

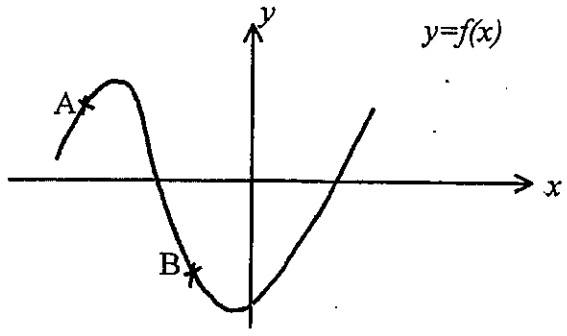
All necessary working should be shown.

START EACH QUESTION ON A NEW PAGE.

Question 1

Marks

(a)



The graph of $y=f(x)$ is shown. Determine the sign of -

(i) $f'(x)$ at A **1**

(ii) $f''(x)$ at B **1**

(b) Find a primitive function of $3x^2 - \frac{1}{3x^2}$ **2**

(c) The gradient at any point (x, y) on a curve is $3x^2 - 2x + 3$. Find the equation of the curve given that it passes through the point $(3, 3)$. **2**

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Question 2 **Marks**

For the curve $y = \dfrac{x^2-1}{x^2-4}$

(a) Find any stationary points and determine their nature. **3**

(b) Determine the equations of any (i) vertical asymptotes
(ii) horizontal asymptotes **3**

(c) Draw a neat sketch of the curve showing its main features, including the intercepts with the axes. **3**

(d) Find the greatest value of the function in the domain $-3 \le x \le 3$ **1**

Question 3

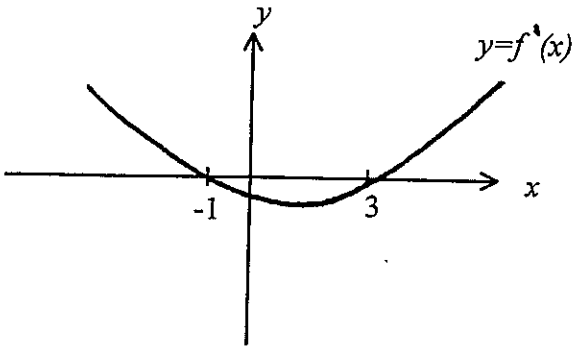
(a) Show that there is a point of inflection at (2, 6) on the curve $y = x^3 - 6x^2 + 8x + 6$ **3**

(b) For a certain function $f'(x) = \dfrac{(x+3)(x-1)^{16}}{\sqrt{x^2 - 4x + 5}}$ **2**

Determine the nature of the stationary point at $x = 1$.

Question 4

(a)

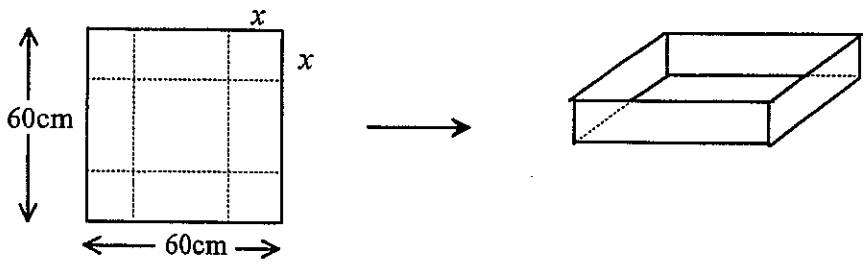


The diagram shows the graph of the derived function $y=f'(x)$ for a given function $y=f(x)$.

Copy the diagram and on it show clearly a possible graph of $y=f(x)$. **3**

Marks

(b)



A square sheet of metal of side 60cm has squares of side $x$cm cut from each corner, as shown above. The edges are then bent up to form an open tray.

(i) Show that the volume of the tray is $V = 3600x - 240x^2 + 4x^3$ **1**

(ii) Hence find the value of x which makes this volume a maximum. **3**

Question 5

(a) (i) How much does \$1200 amount to if it is invested for 20 years at 9% p.a. interest compounded monthly? **1**

(ii) Harry contributes \$1200, at the beginning of every year, into a superannuation fund which pays 9% p.a. interest compounded monthly. How much will be in the fund at the end of the twentieth (20th) year? **3**

(b) Sue borrows \$10,000 from a bank which charges 12% p.a. interest on the balance owing at the end of each month. She agrees to repay the loan in 36 equal monthly repayments.

(i) Find the amount of each monthly repayment. **3**

(ii) Calculate the amount still owing after 1 year, (i.e. after the 12th repayment) **2**

(iii) How much interest was charged for the first year? **2**

Question 6

Prove by mathematical induction that $n^3 - n$ is a multiple of 6 for all positive integers $n \geq 2$. **6**

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Q1

a) (i) +

(ii) [0 ∴ pt. of inflection] but concave up ∴ $f''(x) > 0$

b) $f'(x) = 3x^2 - \frac{x^{-2}}{3}$

$f(x) = x^3 - \frac{1}{3}\frac{x^{-1}}{-1} + c$

$= x^3 + \frac{1}{3x} + c$

c) $y' = 3x^2 - 2x + 3$

$y = x^3 - x^2 + 3x + c$

but $x = 3, y = 3$

$3 = 27 - 9 + 9 + c$

$c = -24$

$y = x^3 - x^2 + 3x - 24$

Q2. $y = \frac{x^2-1}{x^2-4}$

a) $y' = \frac{(x^2-4)(2x) - (x^2-1)(2x)}{(x^2-4)^2}$

$= \frac{2x(x^2 - 4 - x^2 + 1)}{(x^2-4)^2}$

$= \frac{-6x}{(x^2-4)^2}$

for st. pts $y' = 0$

$-6x = 0 \implies x = 0$

x	-1	0	1
y'	+	0	−

/ — \

∴ max at $(0, \frac{1}{4})$

for vertical asymptotes

$x^2 - 4 = 0$

$x = \pm 2$

for horizontal asymptotes.

$\lim_{x \to \pm\infty} \frac{x^2-1}{x^2-4} = \lim_{x \to \pm\infty} \frac{1 - \frac{1}{x^2}}{1 - 4/x^2}$

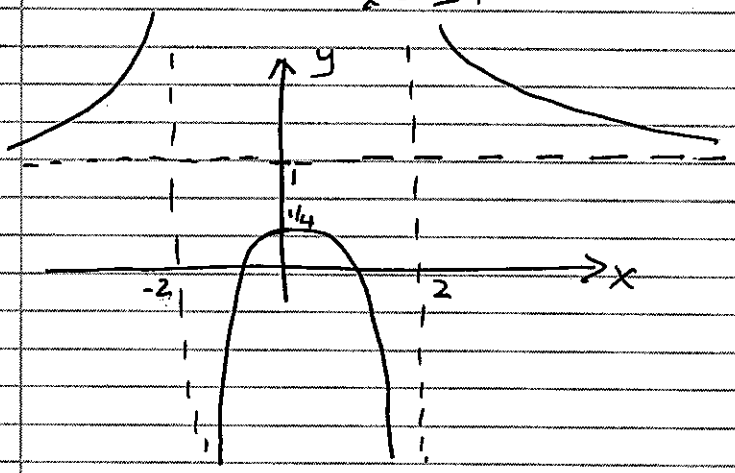
$= 1^+$ $\quad \frac{1 - \frac{1}{100}}{1 - \frac{4}{100}} \to 1$

c) if $x = 0$ $y = \frac{1}{4}$

note $-2 < x < 2$ $\quad y \leq 1/4$.

x intercepts $x^2 - 1 = 0$

$x = \pm 1$



for ~~+~~ $-3 \leq x \leq 3$

max value of function

~~at left pt of $x = -3$~~

~~if $x = -3$ $y = \frac{9-1}{9-4} = \frac{8}{5}$~~

Hence is no max value

as $x \to -2^-$ $y \to +\infty$

$x \to 2^+$ $y \to +\infty$

Q3.

$y = x^3 - 6x^2 + 8x + 6$

$y' = 3x^2 - 12x + 8$

$y'' = 6x - 12$

for inflection $y'' = 0$

$6x - 12 = 0$

$x = 2$ note at $x = 2$

$y' = 12 - 24 + 8$

> 0

$\therefore$ not a max/min

if $x = 2$ $y = 8 - 24 + 16 + 6$

$= 6$

$\therefore (2, 6)$ is a pt. of inflection

b) $f'(x) = \dfrac{(x+3)(x-1)^{16}}{\sqrt{x^2 - 4x + 5}}$

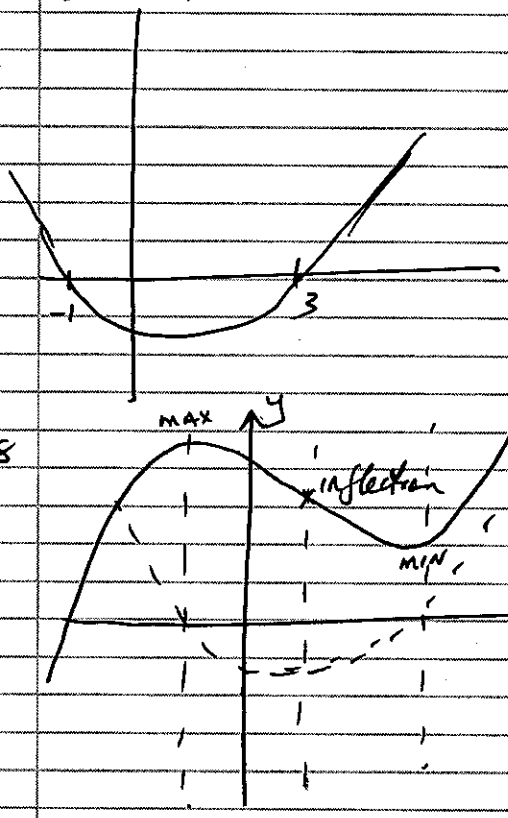
possible st. points at -3 and 1.

x	-4	-3	0	1	2
$f'(x)$	$-$	0	$+$	0	$+$

\ — / — /

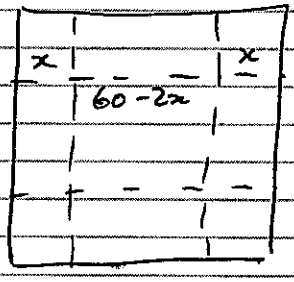
at $x = 1$ there is a horizontal pt. of inflection.

Q4 a)



b)

(i)



note $x < 30$

$V = (60 - 2x)^2 . x$

$= (3600 - 240x + 4x^2)x$

$V = 3600x - 240x^2 + 4x^3$

(ii) for max/min $\dfrac{dV}{dx} = 0$.

$V' = 3600 - 480x + 12x^2$

$= 12(300 - 40x + x^2)$

$= 12(x^2 - 40x + 300)$

$= 12(x - 10)(x - 30)$ but $x < 30$

$= 0$ if $x = 10$

$V'' = -480 + 24x$

if $x = 10$ $V'' = -240 < 0$

∴ max volume if $x = 10$

Q5

20 years = 240 months.

a)(i) 9% pa $= \frac{9}{12}\%$ p/month

$= 0.75\%$.

$\text{Amount} = P\left(1 + \frac{r}{100}\right)^n$

$= 1200(1.0075)^{240}$

$= 7210.981829$

$= \$7210.98$

(ii) $\text{Total} = 1200(1.0075)^{240} + 1200(1.0075)^{228} + 1200(1.0075)^{216} + \vdots + 1200(1.0075)^{12}$ (20 terms.)

$GP\ a = 1200(1.0075)^{12}$

$r = (1.0075)^{12}$ $n = 20$.

$S_{20} = \frac{a(r^{20} - 1)}{r - 1}$

$= \frac{1200(1.0075)^{12}\left[(1.0075)^{240} - 1\right]}{(1.0075)^{12} - 1}$

$= \$70089.23171\ldots$

$= \$70089.23$

Q5b. M = payment / monthly

A_n = money due at end of month n

12% pa $= 1\%$ p month

$A_1 = 10000 \times 1.01$

$A_2 = (10000 \times 1.01 - M) \times 1.01$

$A_3 = (10000 \times 1.01^2 - (M)1.01)1.01$

$= 10000 \times 1.01^3 - M \times 1.01^2$

$A_4 = 10^4 \times 1.01^4 - M \times 1.01^3 - M \times 1.01$

$A_5 = 10^4 \times 1.01^5 - M \times 1.01^4 - M 1.01^2$

Something is wrong.

start again

M = monthly payment

A = Money due after monthly payment

$A_1 = (10000 - M)1.01$

$= 10^4 \times 1.01 - M$

$A_2 = (A_1) \times 1.01 - M$

$= 10^4 \times 1.01^2 - M\,1.01 - M$

$A_3 = A_2 \times 1.01 - M$

$= 10^4 \times 1.01^3 - M(1.01^2 + 1.01 + 1)$

$A_{36} = 10^4 \times 1.01^{36} - M(1.01^{35} + \ldots + 1)$

but $A_{36} = 0$

$M = \frac{10^4 \times 1.01^{36}}{1 + 1.01 + 1.01^2 + \cdots + 1.01^{35}}$

GP $a = 1$, $r = 1.01$, $n = 36$.

$= \frac{10^4 \times 1.01^{36}}{\left[\frac{1 \times 1.01^{36} - 1}{0.01}\right]} = \frac{10^2 \times 1.01^{36}}{(1.01^{36} - 1)}$

$M = 332.143098\ldots$

Monthly repayment $= \$332.14$

5b(ii)

$$A_{12} = 10^4 \times 1.01^{12} - M\left(1 + 1.01 + 1.01^2 + \cdots + 1.01^{11}\right)$$

$$= 10^4 \times 1.01^{12} - 332.14\left(\frac{1\times(1.01^{12}-1)}{.01}\right)$$

$$= 7055.88375\ldots$$

$\therefore$ \$7055.88 still owing after 1 year. ✓

b(iii)

[illegible]

10^4 borrowed

7055.88 owing

2944.12 paid off

but 332.14×12 paid

$= 3985.68$

Interest $= 3985.68 - 2944.12$

$= 1041.56$

$= \$1041.56$

Q6 To show $n^3 - n \mid 6 \quad \forall n \geq 2$

$P(n)$

If $n = 2$ $\quad 8 - 2 = 6$ a multiple of 6

$\therefore P(n)$ True for $n = 2$.

~~Assume true for~~

$P(k)$

$k^3 - k = 6m$

Assume True $P(k)$ True

$P(k+1)$

$(k+1)^3 - (k+1) = (k+1)\left((k+1)^2 - 1\right)$

$= (k+1)(k^2+2k)$

$= k^3 + 2k^2 + k^2 + 2k$

$= k^3 - k + 3k^2 + 3k.$

$= 6m + 3\left[k(k+1)\right]$

Must have a factor of 2 $= 2p.$

$= 6m + 6p.$

$= 6(\quad)$

$\therefore$ has a factor of 6

$P(k)$ True $\Rightarrow P(k+1)$ True

~~Eg~~ True for $n = 2$ $\therefore$ true $n = 3$

$n = 3$ $\therefore$ true $n = 4$

by PMI ~~true~~ for all $n \geq 2$.
