KILLARA HIGH SCHOOL

YEAR 12 MATHEMATICS

EXTENSION 1

TERM 2 ASSESSMENT
2004

Time: 1 hour **40 marks**

* **Approved calculators may be used.**
* **Start each question on a <u>new</u> page.**
* **Show all working.**
* **Marks <u>may not</u> be awarded for untidy work.**

Question 1 (10 marks)

(a) A company assumes that the proportion P of viewers who will buy a new product after it is advertised n times on television satisfies the relation $P = 1 - e^{kn}$, where k is a constant. If 50 per cent of viewers buy the product after 10 advertisements appear, how many times should the company advertise the product if it wants at least 90 per cent of its viewers to buy it? *(5 marks)*

(b) Write down a formula for calculating $\frac{d}{du}F(u)$ when u is a function of x.

Differentiate (with respect of x) $(\tan^{-1}(\frac{x}{3}))^2$

and hence find the exact value of $\frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}(\frac{x}{3})}{x^2+9}dx$ *(5 marks)*

Question 2 (10 marks)

(a) Find the derivative (with respect to x) of:

(i) $\sin^{-1}2x$ for $|x| < \frac{1}{2}$ *(2 marks)*

(ii) $\frac{\sin^{-1}x}{x}$ *(2 marks)*

Question 2 continued:

(b) Find the integrals, with regard to x, of:

(i) $\dfrac{1}{\sqrt{4-x^2}}$ *(1 mark)*

(ii) $\dfrac{-2}{\sqrt{9-4x^2}}$ *(1 mark)*

(c) Find $\int \dfrac{x}{\sqrt{1-9x^4}}dx$ using the substitution $u = 3x^2$ *(4 marks)*

Question 3 (11 marks)

Firefighters are forced to stay 60 metres away from a dangerous fire burning in a low open tank on horizontal ground. They have two pumps. One, which can eject water in any direction at 30 ms^{-1}, is on the ground, while the other, which can eject water at 40 ms^{-1} but only horizontally, is on a vertical stand 5m high. Can both pumps reach the fire? Justify your answer. (Assume that $g = 10ms^{-2}$ and that all frictional forces, including air resistance, can be neglected.)

Question 4 (9 marks)

(a) A spherical bubble is expanding so that its volume is increasing at a constant rate of $10mm^3$ per second. What is the rate of increase of the radius when the surfae area is $500mm^2$? *(3 marks)*

(b) A particle P moves along the x axis according to the rule $x = 4\sin 3t$, where x cm is the displacement of P from O at time t seconds.

(i) Show that P moves in simple harmonic motion and state the period of the motion. *(2 marks)*

(ii) Find when the particle first is $2cm$ to the right of O and find its velocity at that time. *(2 marks)*

(iii) Find the greatest speed of P and the interval in which it moves. *(2 marks)*

END OF PAPER

N.H.S. Extension 1 Term 2 Assessment 2004.

1(a) $P = 1 - e^{kn}$

$0.5 = 1 - e^{k \times 10}$ ✓

$-0.5 = -e^{10k}$

$k = \frac{-0.69}{10}$

5

$= -0.0693$ ✓

$0.9 = 1 - e^{-0.0693 \times n}$ ✓

$-0.1 = -e^{-0.0693n}$

$n = \frac{\ln 0.1}{-0.069..}$ ✓

$= 33.219$

The company should advertise 34 times on TV ✓

(b) $\frac{d}{du} F(u)$

$u = f(x)$

$= \frac{d}{du} F(f(x))$

$\frac{du}{dx} = f'(x)$

$= F'(u) \times f'(x)$

$= \frac{d}{du} F(u) \times \frac{du}{dx}$

✓ either or similar

5

$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2$

$f(x) = \tan^{-1}\left(\frac{x}{3}\right)$

$f'(x) = \frac{3}{3^2+x^2}$

$= 2\left(\tan^{-1}\left(\frac{x}{3}\right)\right)\left(\frac{3}{9+x^2}\right)$ ✓

$= \frac{6\tan^{-1}\frac{x}{3}}{9+x^2} = 6\,\frac{\tan^{-1}\left(\frac{x}{3}\right)}{9+x^2}$

$\therefore \frac{1}{\pi}\int_0^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{x}{3}\right)}{x^2+9}\,dx$

$= \frac{1}{6\pi}\left[\left(\tan^{-1}\left(\frac{x}{3}\right)\right)^2\right]_0^{\sqrt{3}}$ ✓

$= \frac{1}{6\pi}\left\{\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)^2 - \left(\tan^{-1} 0\right)^2\right\}$

$= \frac{1}{6\pi}\left(\frac{\pi^2}{6^2} - 0\right)$ ✓

$= \frac{\pi}{216}$ ✓

2(a)(i) $\frac{d}{dx} \sin^{-1} 2x$

$= \frac{1}{\sqrt{1-(2x)^2}} \times 2$ ✓

2

$= \frac{2}{\sqrt{1-4x^2}}$ ✓

(ii) $\frac{d}{dx} \frac{\sin^{-1} x}{x}$

$= \frac{x\frac{1}{\sqrt{1-x^2}} - (\sin^{-1}x) \times 1}{x^2}$ ✓

2

$= \frac{1}{x\sqrt{1-x^2}} - \frac{\sin^{-1}x}{x^2}$ ✓

(b)(i) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + c$ ✓

1

(ii) $\int \frac{-2}{\sqrt{9-4x^2}}\,dx = -2\int \frac{dx}{\sqrt{9-4x^2}}$

$= -2\int \frac{\frac{1}{2}\,dx}{\sqrt{\frac{9}{4} - x^2}}$ ✓

2

$= -1 \sin^{-1}\frac{x}{\frac{3}{2}} + c$

$= -\sin^{-1}\frac{2x}{3} + c$ ✓

$$2(c)\int \frac{x}{\sqrt{1-9x^4}}\,dx \qquad u = 3x^2$$

$$\frac{du}{dx} = 6x$$

$$= \int \frac{x\,dx}{\sqrt{1-u^2}} \qquad du = 6x\,dx$$

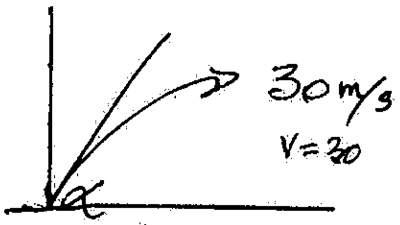
$$\frac{du}{6} = x\,dx$$

4

$$= \frac{1}{6}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{6}\sin^{-1} 3x^2 + c$$

3. Scenario 1



If water finishes on ground level $y=0$.

Vertical motion $\ddot{y} = -g = -10$

$$\dot{y} = \int -g\,dt$$
$$= -10t + c$$

$\dot{y} = V\sin\alpha = -10t + c$

when $t=0$, $\dot{y} = c$

$\therefore \dot{y} = -10t + 30\sin\alpha$

$$y = \int -10t + 30\sin\alpha$$
$$= -\frac{10}{2}t^2 + 30t\sin\alpha + c$$

when $t=0$, $y=0$, $c=0$

$\therefore y = -5t^2 + 30t\sin\alpha$

but $y=0$, $\therefore 0 = -5t(t - 6\sin\alpha)$

$t=0$ or $t = 6\sin\alpha$

Maximum range occurs when $t = 6\sin\alpha$

horizontal motion $\ddot{x} = 0$

$$\dot{x} = \int 0\,dt$$
$$= c$$

when $t=0$, $\dot{x} = V\cos\alpha$

$\therefore V\cos\alpha = c$

when $t=0$, $\dot{x} = V\cos\alpha$

$$x = \int 30\cos\alpha\,dt$$
$$= 30t\cos\alpha + c$$

when $t=0$, $x=0$, $c=0$

$\therefore x = 30t\cos\alpha$

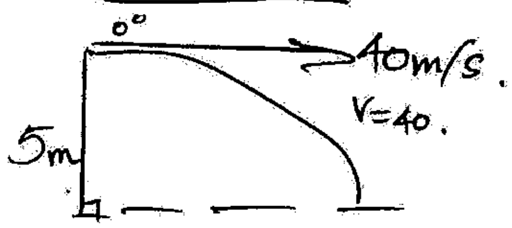
$$= 30 \times 6\sin\alpha\cos\alpha$$
$$= 2\sin\alpha\cos\alpha \times 90$$
$$= 90\sin 2\alpha$$

Maximum for $\sin 2\alpha$ is 1.

$\therefore x = 90$

At 90 metres hose will easily reach!

Scenario 2



Similarly $\dot{x} = 40\cos 0 = 40$

$\therefore x = 40t + c$

when $x=0$, $t=0$, $x = 40t$

$\therefore t = \frac{x}{40}$

$\ddot{y} = -10$

$\dot{y} = -10t + c$

$40\sin 0 = -10t + c$

$c = 0$ at $t=0$.

$\therefore \dot{y} = -10t$

$y = -5t^2 + c$

when $t=0$, $y=5$, $5 = -5\times 0^2 + c$

$\therefore c = 5$

$\therefore y = -5t^2 + 5$

3(contd) Scenario 2:
when $y=0$, water reaches ground.

$0 = -5t^2+5$

$5t^2=5$

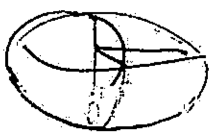
$t^2=1$

$\left(\frac{x}{40}\right)^2=1$

$x^2=40^2$

$x=\pm 40$

∴ the hose will <u>not</u> reach fire!

4.(a) 

$\frac{dV}{dt}=10\,mm^3/s$

$V=\frac{4}{3}\pi r^3$

$\frac{dV}{dr}=4\pi r^2$

$\frac{dr}{dt}=?$

3

$\frac{dV}{dr}\times\frac{dr}{dt}=\frac{dV}{dt}$

$\frac{dV}{dr}\times\frac{dr}{dt}=10$

$4\pi r^2\times\frac{dr}{dt}=10$

$\frac{dr}{dt}=\frac{10}{4\pi r^2}$ but $4\pi r^2=500$

$=\frac{10}{500}$

$=0.02$

Rate of increase in radius when surface area is $500mm^2$ is 0.02 mm / second.

4(b)(i) $x=4\sin 3t$

$\frac{dx}{dt}=12\cos 3t=v$

$\frac{d^2x}{dt^2}=-36\sin 3t=a$

i.e. $a=-9(4\sin 3t)$

$=-3^2x$

3

this satisfies the law of S.H.M. that states $a=-n^2x$ when $n=3$.

The period is $\frac{2\pi}{n}=\frac{2\pi}{3}$ second

(ii)(a) $x=4\sin 3t$ when $x=2$

$2=4\sin 3t$ $\div 4$

$\frac{1}{2}=\sin 3t$

$\sin\frac{\pi}{6}=\frac{1}{2}=\sin 3t$

∴ $3t=\frac{\pi}{6}, \pi-\frac{\pi}{6}, 2\pi+\frac{\pi}{6}, \ldots$

∴ $t=\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}\ldots$

2

Particle is first at $x=2$ at $\frac{\pi}{18}$ seconds

(b) $\cancel{v}=12\cos 3t$

$=12\cos\frac{\pi}{6}$

$=6\sqrt{3}$

1

Velocity at this time is $6\sqrt{3}$ cm/second.

(iii) $v=12\cos 3t$

cos 3t has maximum at 1 i.e. when $v=12$

$\therefore$ 12 metres per second
is greatest speed.

Particle oscillates between
-4_m and $+4_m$ (amplitude)

2