KILLARA HIGH SCHOOL

YEAR 12 MATHEMATICS EXTENSION 1

TERM 1 ASSESSMENT
2004

Time Allowed: 2 hours

72 marks

Approved calculators may be used.
Show ALL necessary working.
Marks <u>may not</u> be awarded for untidy work.

Question 1 (10 marks) **Marks**

(a) Give exact values for

(i) $\sin^{-1}(\frac{1}{\sqrt{2}})$ 1

(ii) $\tan^{-1}(-\frac{1}{\sqrt{3}})$ 1

(b) If $\sin\alpha=\frac{3}{4}$, $0<\alpha<\frac{\pi}{2}$ and $\sin\beta=\frac{2}{3}$, $\frac{\pi}{2}<\beta<\pi$, find the exact values of

(i) $\tan 2\alpha$ 2

(ii) $\cos(\alpha-\beta)$ 3

(c) Find the exact value of $\sin(2\tan^{-1}\frac{1}{\sqrt{5}})$ 3

Question 2 (12 marks) **Marks**

(a) Simplify the following:

(i) $\frac{d}{dx}(\tan x)$ 1

(ii) $\frac{d}{dx}(\tan^2 x)$ 1

(iii) $\int \cos^2 x \ dx$ 2

(iv) $\int \sin x \cos^2 x \ dx$ 3

(v) $\int 5^x \ dx$ 2

(b) Using the substitution $u=1+t$, find $\int \frac{t}{\sqrt{1+t}} dt$ 3

Question 3 (12 marks)

(a) Using the substitution $u=2-x$, evaluate $\int_1^2 x\sqrt{2-x}\, dx$ 5

(b) r and θ vary in such a way that the area of the sector POQ has a constant value of $100cm^2$; show that $\theta=\frac{200}{r^2}$ 2

(c) Using the result $\lim_{x\to 0} \frac{\sin x}{x}=1$, find $\lim_{x\to 0} \frac{1-\cos 2x}{x^2}$ 3

(d) Simplify $\int \frac{6x+12}{x^2+4x} dx$ 2

Question 4 (12 marks)

(a) For the function $y=(x-1)^2-3$

(i) sketch the curve, showing all important features 3

(ii) find the largest possible domain such that this function will have an inverse 2

(b) (i) Find the domain and range of $y=\cos^{-1}2x$ 2

(ii) Sketch the curve in (i). 2

Question 4 continued: **Marks**

(c) For the function $y=\sqrt{2-x}$

(i) find $\frac{dy}{dx}$ **1**

(ii) make x the subject and hence find $\frac{dx}{dy}$. Hence... **1**

(iii) show that $\frac{dy}{dx}\times\frac{dx}{dy}=1$ **1**

Question 5 (12 marks)

(a) (i) Evaluate $\dfrac{3^{\pi}-\ln 3}{\sqrt{\pi}+e}$ to 3 significant figures. **2**

(ii) Express as a single logarithm: $5\log x+3\log y-\frac{1}{3}\log z$ **2**

(b) Differentiate (i) e^{6x+3} **1**

(ii) $e^{2x^3}+\log 3x$ **2**

(iii) $5x-\log 2x^4$ **2**

(iv) $\ln\dfrac{e^x+1}{e^x-1}$ **3**

Question 6 (13 marks)

(a) (i) Solve $2\cos^2\theta+3\cos\theta-2=0$ for $0\le\theta\le 2\pi$ **2**

(ii) What is the general solution of this equation in (i)? **1**

(b) Write down the range of values for $y=e^{\sin x}$ and find the smallest possible positive value of x for which it is to be a minimum. **3**

(c) Show that $\dfrac{x+7}{x-1}$ can be written as $1+\dfrac{8}{x-1}$ and hence show that $\int_2^{e+1}\dfrac{x+7}{x-1}dx=e+7$. **3**

(d) The arc of the curve $y=\log_e x$ from $x=1$ to $x=e$ is rotated about the y axis. Find the volume of the resultant solid. **4**

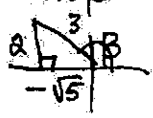
END OF PAPER

2004 Term 1 Worked Solutions
Year 12 Extension 1

1. (a)(i) $\sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$ ✓ 45°?

(ii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ ✓ -30°?

(b) $\sin\alpha = \frac{3}{4}$ $\quad \sin\beta = \frac{2}{3}$



(i) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$= \dfrac{2\times\frac{3}{\sqrt{7}}}{1-\frac{9}{7}}$ ✓

$= \frac{6}{\sqrt{7}} \div -\frac{2}{7}$

$= -\frac{42}{2\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}$

$= -\frac{42\sqrt{7}}{2\times 7}$

$= -3\sqrt{7}$ ✓

(ii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ✓

$= \frac{\sqrt{7}}{4}\times -\frac{\sqrt{5}}{3} + \frac{3}{4}\times\frac{2}{3}$ ✓

$= \frac{-\sqrt{35}}{12} + \frac{1}{2}$ ✓

$= \frac{6-\sqrt{35}}{12}$

(c) $\sin\left(2\tan^{-1}\frac{1}{\sqrt{5}}\right)$

✓

$= \sin 2\theta$

$= 2\sin\theta\cos\theta$ ✓

$= 2\times\frac{1}{\sqrt{6}}\times\frac{\sqrt{5}}{\sqrt{6}}$ ✓

$= \frac{2\sqrt{5}}{6}$

$= \frac{\sqrt{5}}{3}$ //

10

2 (i) $\frac{d}{dx}(\tan x) = \sec^2 x$ ✓

(ii) $\frac{d}{dx}(\tan^2 x) = \sec^2 x \times 2\tan x$

$= 2\tan x\sec^2 x$ ✓

(iii) $\int\cos^2 x\,dx = \frac{1}{2}\int(\cos 2x+1)\,dx$ ✓

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x + c\right)$

$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$ ✓

(iv) $y = \cos^3 x$

$y' = -3\cos^2 x \times \sin x$ ✓

$\therefore \int \sin x\cos^2 x\,dx$

$= -\frac{1}{3}\int -3\sin x\cos^2 x\,dx$ ✓

$= -\frac{1}{3}\cos^3 x + c$ ✓

(v) $\int 5^x\,dx \qquad 5 = e^{\ln 5}$

$= \frac{1}{\ln 5}\times 5^x + c$ ✓

$= \frac{5^x}{\ln 5} + c$ ✓

12

$$2(b)\ \int \frac{t}{\sqrt{1+t}}\,dt \qquad u = 1+t,\quad \frac{du}{dt} = 1,\quad dt = du,\quad t = u-1$$

$$= \int \frac{t}{u^{1/2}}\,dt$$

$$= \int \frac{u-1}{u^{1/2}}\,du$$

$$= \int (u^{1/2} - u^{-1/2})\,du$$

$$= \frac{2}{3}u^{3/2} - 2u^{1/2} + c$$

$$= \frac{2}{3}(1+t)^{3/2} - 2(1+t)^{1/2} + c$$

$$3.(a)\ \int_1^2 x\sqrt{2-x}\,dx \qquad u = 2-x,\quad x = 2-u,\quad \frac{du}{dx} = -1,\quad du = -dx$$

$$= \int_1^2 (2-u)\,u^{1/2}(-du)$$

$$= -\int_1^0 (2u^{1/2} - u^{3/2})\,du \qquad u = 2-2 = 0,\quad u = 2-1 = 1$$

$$= \int_0^1 (2u^{1/2} - u^{3/2})\,du$$

$$= \left[\frac{2u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_0^1$$

$$= \left[\frac{4u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$$

$$= \frac{4}{3} - \frac{2}{5} - (0)$$

$$= \frac{14}{15}.$$

(b) Area Sector $= \frac{1}{2}r^2\theta_c$

$$100 = \frac{1}{2}r^2\theta_c$$

$$\frac{200}{r^2} = \theta$$

$$\text{i.e. } \theta = \frac{200}{r^2}$$

$$3(c)\ \lim_{x\to 0} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{x\to 0} \frac{1-(1-2\sin^2 x)}{x^2}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$$

$$= 2\lim_{x\to 0} \frac{\sin^2 x}{x^2}$$

$$= 2\lim_{x\to 0} \frac{\sin x}{x} \times \frac{\sin x}{x}$$

$$= 2\times 1\times 1$$

$$= 2.$$

12

$$(d)\ \int \frac{6x+12}{x^2+4x}\,dx$$

$$= 3\int \frac{2x+4}{x^2+4x}\,dx$$

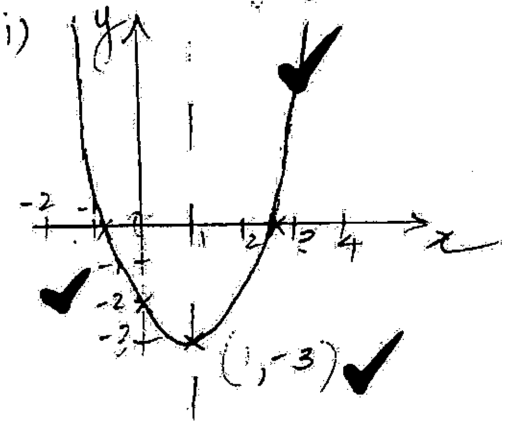
$$= 3\ln(x^2+4x) + c$$

4.(a) $y = (x-1)^2 - 3$ $\quad x = 0,\ y = -2$

vertex $(1,-3)$

axis of symmetry $x = 1$

(i)

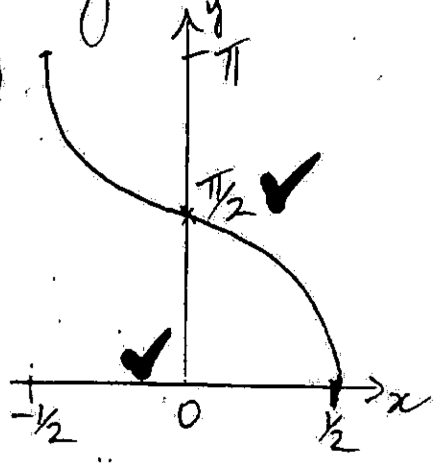


(ii) $x \leqslant 1$

or $x \geqslant 1$

7 (b) $y = \cos^{-1} 2x$

(ii)



domain $-\frac{1}{2} \leq x \leq \frac{1}{2}$

range $0 \leq y \leq \pi$

(c) $y = \sqrt{2-x} = (2-x)^{1/2}$

(i) $\frac{dy}{dx} = \frac{1}{2}(2-x)^{-1/2} \times -1$

$= \frac{-1}{2\sqrt{2-x}}$

(ii) $y^2 = 2-x$

$x = 2-y^2$

$\frac{dx}{dy} = -2y$

(iii) $LHS = \frac{dy}{dx} \times \frac{dx}{dy}$

$= \frac{-1}{2\sqrt{2-x}} \times -2y$

$= -\frac{1}{2\sqrt{2-x}} \times -2\sqrt{2-x}$

$= +1$

$= RHS$

12

5 (a)(i) $\frac{3^{\pi} - \ln 3}{\sqrt{\pi} + e}$

$= \frac{30.44}{4.49}$

5 (a)(ii) $5 \log x + 3 \log y - \frac{1}{3} \log z$

$= \log x^5 + \log y^3 - \log z^{1/3}$

$= \log \frac{x^5 y^3}{\sqrt[3]{z}}$

(b) (i) $\frac{d}{dx}\left(e^{6x+3}\right)$

$= 6e^{6x+3}$

(ii) $\frac{d}{dx}\left(e^{2x^3} + \log 3x\right)$

$= 6x^2 e^{2x^3} + \frac{3}{3x}$

$= 6x^2 e^{2x^3} + \frac{1}{x}$

(iii) $\frac{d}{dx}\left(5x - \log 2x^4\right)$

$= 5 - \frac{8x^3}{2x^4}$

$= 5 - \frac{4}{x}$

(iv) $\frac{d}{dx}\left(\ln \frac{e^x+1}{e^x-1}\right)$

$= \frac{d}{dx}\left(\ln(e^x+1) - \ln(e^x-1)\right)$

$= \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1}$

$= \frac{-2e^x}{e^{2x}-1}$

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6(a)(i) $2\cos^2\theta + 3\cos\theta - 2 = 0$

$2\cos^2\theta + 4\cos\theta - \cos\theta - 2 = 0$

$2\cos\theta(\cos\theta + 2) - (\cos\theta + 2) = 0$

$(\cos\theta + 2)(2\cos\theta - 1) = 0$

$\cos\theta = -2$ no solution ✓ $\quad 2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}$ ✓

(ii) $\theta = 2\pi n \pm \cos^{-1}\frac{1}{2}$ ✓ $= 2n\pi \pm \frac{\pi}{3}$

(b) $-1 \leq \sin x \leq 1$

$y = e^{\sin x}$ $\quad \therefore \frac{1}{e} \leq y \leq e$ ✓

$y' = e^{\sin x} \cdot \cos x$

minimum at $y' = 0$

$\therefore 0 = \cos x \, e^{\sin x}$

$\cos x = 0, \; x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2} \ldots$

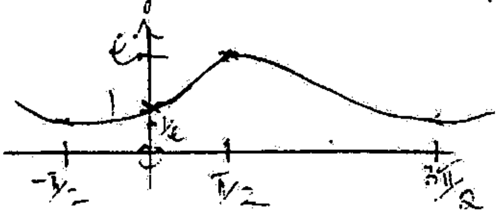
$x =$	$\frac{\pi}{2}^-$	$\frac{\pi}{2}$	$\frac{\pi}{2}^+$
$y' =$	$++$	0	$+-$
	/	—	\

$\therefore$ maximum

$x =$	$-\frac{\pi}{2}^-$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}^+$
y'	$+-$	0	$++$
	\	—	/

✓

$\therefore$ minimum



$x = \frac{3\pi}{2}$ ✓

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6(c) $1 + \frac{8}{x-1}$

$= \frac{x - 1 + 8}{x-1}$

$= \frac{x+7}{x-1}$ ✓

$\therefore \int_2^{e+1} \frac{x+7}{x-1}\,dx$

$= \int_2^{e+1} \left(1 + \frac{8}{x-1}\right) dx$

$= \left[x + 8\ln(x-1)\right]_2^{e+1}$ ✓

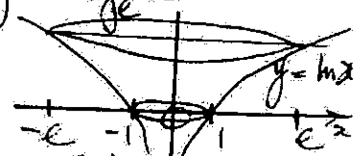
$= e + 1 + 8\ln(e+1-1) - (2 + 8\ln(2-1))$

$= e + 1 + 8\log_e e - (2 + 8\log_e 1)$

$= e + 1 + 8 - (2 + 0)$

$= e + 7$ ✓

6(d) $y = \log_e x$



$V = \pi \int_0^1 x^2\,dy$

$y = \log_e x \quad x = e, \; y = 1$

$e^y = x$ ✓ $\quad x = 1, \; y = 0$ ✓

$V = \pi \int_0^1 e^{2y}\,dy$

$= \pi \left[\frac{1}{2} e^{2y}\right]_0^1$ ✓

$= \frac{\pi}{2}\left(e^2 - e^0\right)$

$= \frac{\pi}{2}(e^2 - 1)\ u^3$ ✓