KILLARA HIGH SCHOOL

YEAR 11 EXAMINATIONS

TERM 4 2005 (HSC)

# MATHEMATICS EXTENSION 1

**Time Allowed:** **70 minutes**

***Instructions:***

* **Attempt ALL questions**
* **Show all necessary working**
* **Marks may NOT be awarded for careless or inadequate working**
* **Board approved calculators may be used**
* **Start each question on a new page**

**Question 1 - Application of Series** (12 marks)

(a) Kim was born on 17th August 1975. On this day his father deposited \$500 in an account earning 9% pa interest. On each subsequent birthday Kim's father deposited another \$500. If Kim's father withdrew all the money in the account the day before Kim's 21st birthday and gave it to Kim as a gift, how much did Kim receive?

(b) A ball is dropped from a height of 8m and bounces to one third of its previous height at each bounce. What is the maximum distance through which the ball would bounce before coming to rest?

**Question 2 - Locus and Parabola** (12 marks)

(a) The chord of the parabola $x^2=4ay$ joining $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meets the focal chord parallel to the $x$-axis at L. Show that the coordinates of L are $\left(\dfrac{2a[1+pq]}{p+q}, a\right)$

(b) The normals to the parabola $x^2=4ay$ at the points $T(2at, at^2)$ and $S(2as, as^2)$ meet at R.

(i) Show that the coordinates of the point R are $(-ats[t+s], a[t^2+ts+s^2+2])$.

(ii) If $st=-2$ find the cartesian equation of the locus of R.

**Question 3 - Geometric Applications of Calculus** (16 marks)

(a) Given $\frac{d^2s}{dt^2}=3$ and that when $\frac{ds}{dt}=5$, $t=0$ and $s=0$, find $s$ in terms of $t$.

(b) In a book the pages are to have an area of 338cm$^2$. A margin of 1cm is left at each side and 2cm at the top and bottom of each page. If the width of the page is $x$ cm show that the area A cm$^2$ of the print is given by $A=346-4x-\frac{676}{x}$. Hence find the dimensions of the page so that there is a maximum area of print.

(c) For the curve $y=x^3+3x^2-9x-8$

(i) Find the stationary points and distinguish between them.

(ii) Find the point of inflexion.

(iii) Sketch the curve.

**Question 4 - Induction** (8 marks)

(a) Use induction to prove that $3^{3n}+2^{n+2}$ is divisible by 5, for positive integral values of $n$.

(b) Prove by induction, for positive integral values of $n$ that

$$1+3+7+15+\ldots\ldots\ldots+(2^n-1)=2^{n+1}-n-2.$$

# Question 1

(a) $R = 9\%$ $\quad A_0 = \$500 \ldots A_{20} = \$500 \quad n = 21$ 1

@ 21 $\quad A_0 = 500(1.09)^{21} = T_{21}$

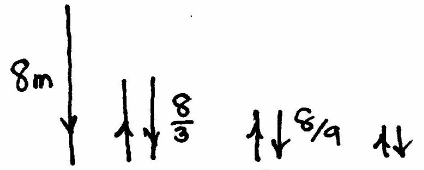
$\quad A_{20} = 500(1.09) = T_1$ } 1

$S_{21} = \underbrace{500(1.09)}_{1}\left[\frac{(1.09)^{21}-1}{1.09-1}\right]$ 1

Kim receives $= \$30\,936.67$ 1

1 mk units ie \$ and • nearest cen

(b)




$a = 8 \quad r = \frac{1}{3} \qquad S_\infty = \frac{8}{1-1/3}$ 1

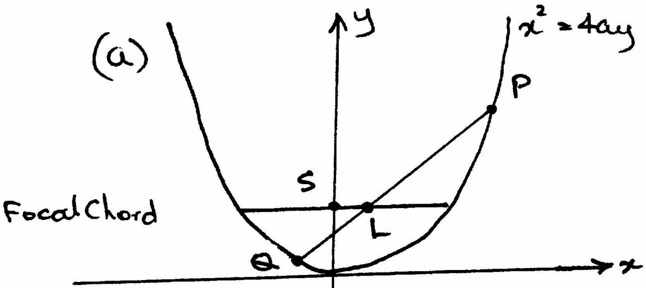
$\quad\ 1 \qquad\quad 1 \qquad\qquad = 12\text{m.}$ 1

Distance travelled $= 2 \times 12 - 8$ 1

$= 16\text{m.}$ 1

# Question 2 Locus and Parabola

(a)




$P(2ap \ \ ap^2) \quad Q(2aq \ \ aq^2)$

Chord $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$

$\dfrac{y-ap^2}{x-2ap} = \dfrac{aq^2-ap^2}{2aq-2ap}$.

$\dfrac{y-ap^2}{x-2ap} = \dfrac{\cancel{(q-p)}(q+p)}{2\cancel{(q-p)}}$

Focus $(0,a)$ $\therefore$ Chord $y = a$ 1

@ L $y = a$

1 $\therefore y - ap^2 = \frac{1}{2}(q+p)(x-2ap)$

$y = \left(\frac{p+q}{2}\right)x - apq - ap^2 + ap^2$

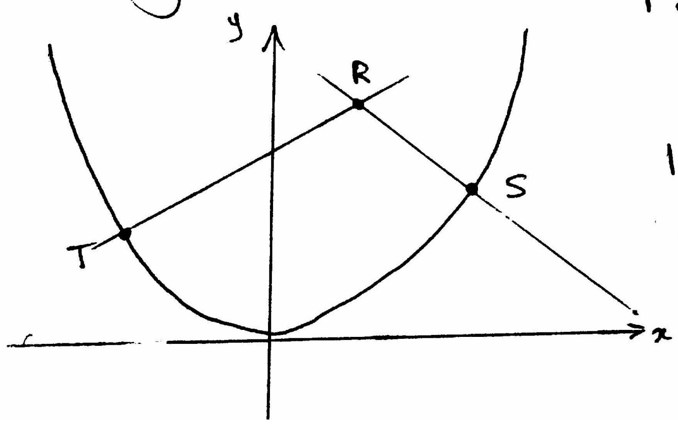
$y = \left(\frac{p+q}{2}\right)x - apq$

1 @ $y = a$ $\quad a + apq = \left(\frac{p+q}{2}\right)x$

$2a(1+pq) = (p+q)x$

1 $\quad x = \dfrac{2a(1+pq)}{p+q}$

$\therefore$ L is $\left(\dfrac{2a[1+pq]}{p+q}, a\right)$



(b) $x^2 = 4ay$ $\quad T(2at, at^2)$ $\quad S = (2as, as^2)$

Tangent @ T
$$y = \frac{x^2}{4a}$$
$$y' = \frac{2x}{4a}$$
$$= \frac{x}{2a}$$
$$\therefore -\frac{1}{m} = -\frac{2a}{x}$$
@ $f'(2at) = -\frac{1}{t}$

1 Deriving

$$y - at^2 = -\frac{1}{t}(x - 2at)$$
$$ty - at^3 = -x + 2at$$
$$x + ty = at^3 + 2at \quad (1)$$
@ S $\quad x + sy = as^3 + 2as \quad (2)$ $\quad$ 1

(1)-(2) $\quad y(\cancel{t-s}) = a(t^3 - s^3) + 2a(\cancel{t-s})$
$$y = a(t^2 + ts + s^2 + 2) \quad 1$$
Sub in (1)
$$x + at(\cancel{t^2} + ts + s^2 + \cancel{2}) = \cancel{at^3} + \cancel{2at}$$
$$x = -ats(t+s) \quad 1$$
$\therefore$ R is $(-ats[t+s],\ a(t^2 + ts + s^2 + 2))$

2 (b) (ii) $\quad x = ats(t+s)$
$$y = a(t^2 + ts + s^2 + 2)$$
If $ts = -2$ $\quad \therefore x = -2a(t+s)$
$$(t+s) = -\frac{x}{2a} \quad 1$$
$$\therefore y = a[(t+s)^2 - ts + 2]$$
$$y = a\left[\left(-\frac{x}{2a}\right)^2 + 2 + 2\right] \quad 1$$
$$y = \frac{x^2 \times a}{4a^2} + 4a \quad 1$$
$$y = \frac{x^2}{4a} + 4a$$
or $4ay = x^2 + 16a^2$
or $x^2 = 4a(y - 4a)$ $\quad$ 1

# QUESTION 3.

(a) $$\frac{d^2s}{dt^2} = 3$$

$$\therefore \frac{ds}{dt} = 3t + k.$$

But $5 = 0 + k$

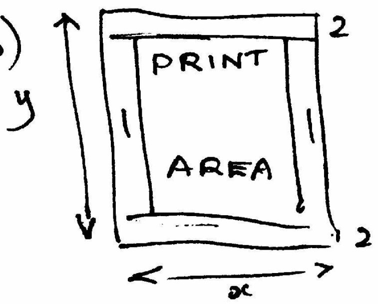
$$\therefore k = 5$$

So $ds/dt = 3t + 5$

$$s = \tfrac{3}{2}t^2 + 5t + c$$

$$0 = 0 + 0 + c$$

$$\therefore s = 3/2\, t^2 + 5t$$

(b) 


$A = 338 = x \times y$

$\therefore y = 338/x.$

Print Area $= (x-2) \times (y-4)$

$= (x-2)\left(\frac{338}{x} - 4\right)$

$= 338 - 4x - \frac{676}{x} + 8$

$A = 346 - 4x - \frac{676}{x}$

$\therefore A' = -4 + \frac{676}{x^2}$

Let $A' = 0$

$\therefore 4x^2 = 676$

$x^2 = 169$

$x = 13$

$y = \frac{338}{13}$

$= 26$

Dimensions 13cm × 26cm.

# QUESTION 3.

7 (c) $y = x^3 + 3x^2 - 9x - 8$

(i) $y' = 3x^2 + 6x - 9$

Let $y' = 0$. $\therefore x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$ 1

$x = -3$ or $x = 1$ 1

$y'' = 6x + 6$

Let $y'' = 0 \quad \therefore x = -1$

$\left.\begin{array}{l} f''(-3) = -18 + 6 < 0 \quad \therefore \text{max} \\ f''(1) = 6 + 6 > 0 \quad \therefore \text{min.} \end{array}\right\}$ 1

$f(-3) = -27 + 27 + 27 - 8 = 19$

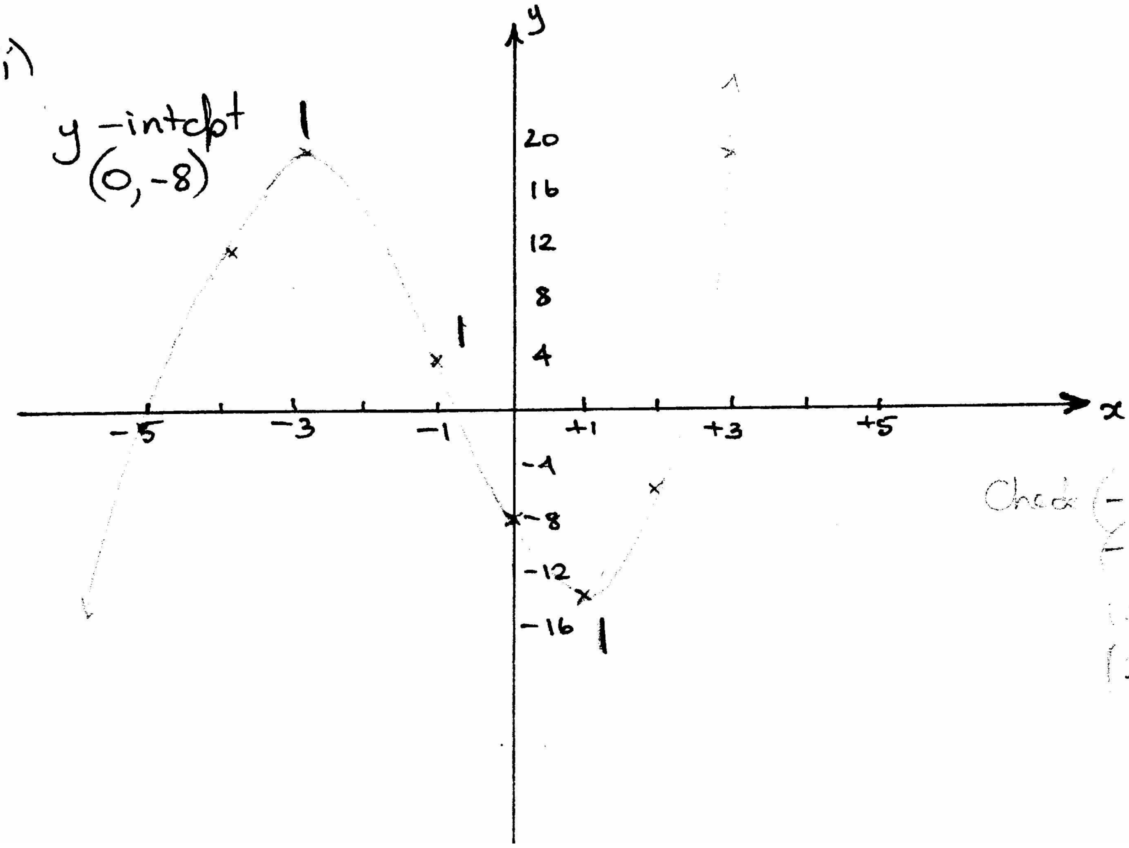
$f(1) = 1 + 3 - 9 - 8 = -13$

$f(-1) = -1 + 3 + 10 - 8 = 4$

$\therefore$ Min @ $(1, -13)$ Max @ $(-3, 19)$

(ii) Point of inflexion $(-1, 4)$. 1

(iii)




Check $(-4, 12)$

$(-5, -1)$

$(3, 19)$

$(2, -6)$

QUESTION 4

(a) $3^{3n} + 2^{n+2}$ divisible by 5

1 Step 1 For $n=1$ LHS $= 3^3 + 2^3$

$= 27 + 8$

$= 35$

$= 5 \times 9$

$\therefore$ True for $n=1$.

1 Step 2 Assume $n = k$.

i.e. $3^{3k} + 2^{k+2} = 5 \times m$. ($m$ an integer)

1 Step 3 Prove for $n = k+1$

i.e. $3^{3(k+1)} + 2^{(k+1)+2} = 5p$. ($p$ an integer

LHS $= 3^{3k+3} + 2^{k+3}$

$= (5m - 2^{k+2}) \times 27 + 2^{k+3}$

$= 5 \times 27m - 2^k \times 4 \times 27 - 2^k \times 8$

$= 5 \times 27m - 2^k(100)$

$= 5 \times (27m - 20 \times 2^k)$

$\therefore \div 5$ or $= 5p$.

1 Step 4 $\therefore$ If true for $n=k$ then true for $n=k$

But true for $n=1$ $\therefore$ true for $n=2$

Now true for $n=2$ $\therefore$ " " $n=3$

and so on for all integer $n \geq 1$.

(b) $1 + 3 + 7 + 15 + \ldots + (2^n - 1) = 2^{n+1} - n - 2$.

1 Step 1 For $n=1$ LHS $=1$ RHS $= 2^2 - 1 - 2 = 1 =$ LHS

$\therefore$ True for $n=1$.

1 Step 2 Assume for $n=k$

i.e. $1 + 3 + 7 + 15 + \ldots + (2^k - 1) = 2^{k+1} - k - 2$

1 Step 3 Prove for $n = k+1$

i.e. $1 + 3 + 7 + 15 + \ldots + (2^{k+1} - 1) = 2^{k+2} - (k+1) -$

$= 2^{k+2} - k$

LHS $= (1 + 3 + 7 + \ldots + 2^k - 1) + (2^{k+1} - 1)$

$= 2^{k+1} - k - 2 + 2^{k+1} - 1$

$= 2 \times 2^{k+1} - k - 3$

$=$ RHS

1 Step 4 $\therefore$ If true for $n=k$ then true for $n=$

But true for $n=1$ $\therefore$ true for $n=$

and so on for all integers $n$.