KILLARA HIGH SCHOOL

YEAR 12 TERM 1 ASSESSMENT 2007

EXTENSION 1 MATHEMATICS

Time Allowed: **2 hours**

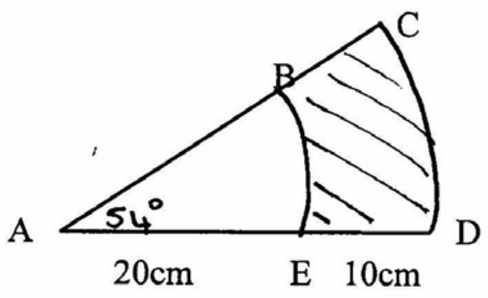
Total marks: **84**

* All questions are of equal value.
* Start each question in a new booklet.
* Write on one side of the paper only.
* Show all necessary working.

Question 1 (12 marks) **Marks**

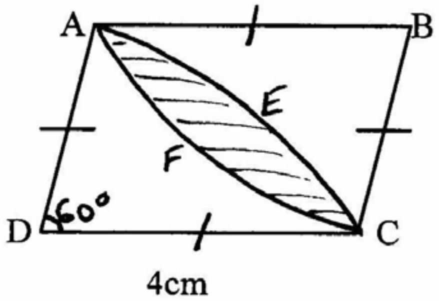
(a) (i) Convert 54° to radians (leave answer in terms of π). **1**

(ii)



CD and BE are arcs of circles with centre A.
AE = 20cm, ED = 10cm.
Find the perimeter of the shaded region BCDE. **2**

(b)



ABCD is a rhombus of side length 4cm. AEC and AFC are arcs of circles with centres B and D respectively.

(i) Show that the area of the rhombus ABCD is $8\sqrt{3}$ cm^2. **1**

(ii) Find the area of the shaded region. **2**

(c) Evaluate $\lim_{x \to 0} \dfrac{\sin 3x}{4x}$ **1**

(d) Differentiate (i) $x \tan x$ **1**

(ii) $\sin^3 x$ **2**

(e) By expressing $\cot x$ in terms of $\sin x$ and $\cos x$, or otherwise, show that the derivative of $\cot x$ is $-\operatorname{cosec}^2 x$. **2**

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Question 2 **(12 marks)** **Marks**

(a) Find primitive functions of

(i) $\sin(4x-3)$ **1**

(ii) $\tan^2 x$ **2**

(iii) $\cos^2 x$ **2**

(b) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos 2x \quad dx$ **2**

(c) A particle is moving in Simple Harmonic Motion. Its displacement x metres at time t seconds is given by $x=2\cos(3t+\frac{\pi}{5})$.

(i) What is the period and amplitude of the motion? **2**

(ii) Show that $\ddot{x}=-9x$ **2**

(iii) Find the maximum speed of the particle. **1**

Question 3. (12 marks) **Marks**

(a) The acceleration, a m/s^2, of a particle moving in a straight line is given by $a = -e^{-2x}$ where x metres is the displacement from the origin at time t seconds. Initially the particle is at the origin with velocity 1m/s. Find the displacement as a function of time. **5**

(b) A projectile is fired with initial speed V m/s at an angle α to the horizontal.

(i) Given that the projectile's position is given by $x = Vt\cos\alpha,\ \ y = Vt\sin\alpha - \frac{1}{2}gt^2$ (you do not need to prove this), show that the equation of the path of the projectile is **3**

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

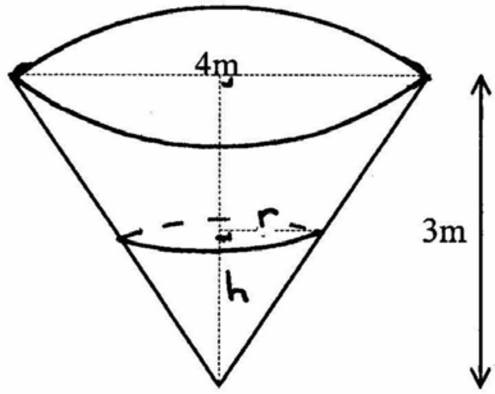
(ii) Show that the range is given by $x = \dfrac{V^2\sin 2\alpha}{g}$ **2**

(iii) Find the maximum range. **1**

(iv) Explain the effect on the maximum range if the initial speed is doubled, all other factors remaining the same. **1**

Question 4 (12 marks) **Marks**

(a)



Water is pouring into a large storage tank which is conical in shape. The diameter of the tank at its top is 4m and its height is 3m. When the height of water is *h* m the radius of the tank is *r* m.

(i) Use similar triangles to show that $r = \dfrac{2h}{3}$ **1**

(ii) Hence show that the volume, V, of water in the tank can be expressed as $V = \dfrac{4\pi h^3}{27}$ **1**

(iii) If water is flowing into the tank at a constant rate of 0.008 m^3/s, show that the height of water in the tank is increasing at the rate $\dfrac{dh}{dt} = \dfrac{0.018}{\pi h^2}$ **2**

(iv) At what rate is the water level rising when the tank is half full? **1**

(b) The rate at which a substance cools is proportional to the difference between its temperature and the temperature of the surrounds, ie $\dfrac{dT}{dt} = k(T - S)$ where T is the temperature of the substance, S is the temperature of the surrounds, *t* is the time in minutes and *k* is a constant.

(i) Show that $T = S + Ae^{kt}$, where A is a constant, satisfies the equation. **1**

(ii) On a day when the temperature is 28°C, a substance cools from 100°C to 80°C in 15 minutes. What will be the temperature of the substance after a further 15 minutes? **4**

(iii) Sketch a graph of temperature T at time *t*, showing its main features. **2**

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Question 5. (12 marks) **Marks**

(a) In how many ways can 6 people be arranged

(i) in a straight line? **1**

(ii) in a circle? **1**

(b) There are 7 swimmers in a race. In how many ways can 1st, 2nd and 3rd place be filled? **1**

(c) Simplify $\dfrac{^nP_5}{^nC_3}$ **3**

(d) How many arrangements of the letters of the word KILLARA are possible? **2**

(e) A committee of 4 people is to be chosen from 9 members of a social club. Bob and Jenny are 2 of the 9 members. How many different committees is it possible to form

(i) if there are no restrictions **1**

(ii) containing both Bob and Jenny **1**

(iii) containing neither Bob nor Jenny **1**

(iv) containing at least one of Bob and Jenny **1**

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Question 6. (12 marks) **Marks**

(a) Find the quotient and remainder when $P(x) = x^4 - 3x^3 + 7x^2 - 5x - 2$ is divided by $A(x) = x^2 + 1$. **3**

(b) The polynomial $P(x) = ax^3 + bx^2 - 8x - 28$ has a zero at -2 and leaves a remainder of -27 when divided by $x - 1$. Find the values of a and b. **3**

(c) The polynomial equation $px^3 + qx^2 + rx + \quad = 0$ has roots which are in the ratio 2 : 3 : 4. Show that $4qr = 39ps$ **4**

(d) Draw a neat sketch, showing the nature of the curve at the x-intercepts, of $y = x^3(x-2)^2$ **2**

Question 7. (12 marks) **Marks**

(a) Differentiate

(i) $e^{\cos x}$ **1**

(ii) $\log_e \sin x$ **1**

(b) (i) Differentiate $\ln(\ln x)$ **1**

(ii) Hence evaluate $\int_e^{e^2} \frac{dx}{x\ln x}$ **3**

(c) Find the constant K such that $y = e^x \cos x$ satisfies
$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + ky = 0$ **3**

(d) Use the substitution $u = \tan x$ to find $\int \frac{\sin^3 x}{\cos^5 x} dx$ **3**

END OF PAPER

1 a) i) $54^\circ = \frac{54\pi}{180}^c = \frac{3\pi}{10}^c$ ✓

) $BE = 20 \times \frac{3\pi}{10}$, $CD = 30 \times \frac{3\pi}{10}$ ✓

erimeter $= 6\pi + 9\pi + 20$ cm

($= 67.12$ cm.)

) (i) $A = (\frac{1}{2} \times 4 \times 4 \times \sin 60^\circ) \times 2$ ✓

$= 8\sqrt{3}$ cm^2

) Sector DAEC $= \frac{1}{2} \times 4^2 \times \frac{\pi}{3}$

$= \frac{8\pi}{3}$ cm^2 ✓

Segment AEC $= \frac{8\pi}{3} - \frac{1}{2} \times 4^2 \times \sin 60^\circ$

$= \frac{8\pi}{3} - 4\sqrt{3}$

Shaded area $= 2 \times$ Segment AEC

$= \frac{16\pi}{3} - 8\sqrt{3}$ cm^2

($= 2.899$)

) $= \frac{3}{4} \times \lim_{3x \to 0} \frac{\sin 3x}{3x}$

$= \frac{3}{4}$ ✓

) i) $x \sec^2 x + \tan x$ ✓

ii) $3 \sin^2 x \cos x$ ✓✓

) $\frac{d}{dx} \frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$ ✓

$= \frac{-1}{\sin^2 x}$ ✓

$= -\csc^2 x$

Q.2 a) (i) $-\frac{1}{4}\cos(4x-3) + c$ ✓

ii) $\int \tan^2 x . dx = \int (\sec^2 x - 1) dx$ ✓

$= \tan x - x + c$ ✓

iii) $\int \cos^2 x . dx = \int (\frac{1}{2} + \frac{1}{2}\cos 2x) dx$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$ ✓

b) $\frac{1}{2}\left[\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ ✓

$= \frac{1}{2}\left[\sin \frac{2\pi}{3} - \sin \frac{\pi}{2}\right]$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$ ✓

c) (i) $T = \frac{2\pi}{3}$ ✓, $a = 2$ ✓

(ii) $\dot{x} = -6\sin(3t + \frac{\pi}{5})$

$\ddot{x} = -18\cos(3t + \frac{\pi}{5})$ ✓

$= -9\left(2\cos(3t + \frac{\pi}{5})\right)$ ✓

$= -9x$

(iii) Max speed $= 6$ ✓

3 (a) $a = -e^{-2x}$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + c$ ✓

$v = 1$ when $x = 0 \Rightarrow \frac{1}{2} = \frac{1}{2} + c$ ✓

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$

$v^2 = e^{-2x}$

$v = \pm e^{-x}$ ✓

Since initially v is moving to the right & v is never zero, take $v = e^{-x}$

$\therefore \frac{dt}{dx} = e^{x}$

$t = e^{x} + c$ ✓

When $t = 0, x = 0 \Rightarrow 0 = 1 + c$

$\therefore t = e^{x} - 1$

$e^{x} = t + 1$

$x = \ln(t+1)$ ✓

) (i) Subst. $t = \frac{x}{V\cos\alpha}$ into ✓

$y = Vt\sin\alpha - \frac{1}{2}gt^2$

$= \cancel{V}\sin\alpha . \frac{x}{\cancel{V}\cos\alpha} - \frac{g}{2} \times \frac{x^2}{V^2\cos^2\alpha}$ ✓

$= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$ ✓

(ii) Find x when $y = 0$

$x\left(\tan\alpha - \frac{gx}{2V^2\cos^2\alpha}\right) = 0$

$x = 0$ or

$\frac{gx}{2V^2\cos^2\alpha} = \tan\alpha$ ✓

$x = \frac{2V^2\cos^2\alpha\tan\alpha}{g}$

$= \frac{2V^2\cos\alpha\sin\alpha}{g}$

$= \frac{V^2\sin 2\alpha}{g}$ ✓

(iii) Max. range $= \frac{V^2}{g}$ ✓

(iv) If V is doubled,

max. range $= \frac{(2V)^2}{g}$

$= 4\frac{V^2}{g}$ ✓

ie. Max. range is quadrupled.

OR (ii) Put $y = 0 \Rightarrow$

$Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(V\sin\alpha - \frac{1}{2}gt) = 0$

$t = 0$, $t = \frac{2V\sin\alpha}{g}$ ✓

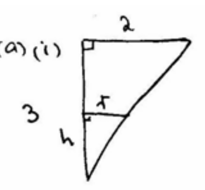
Subst. into $x = Vt\cos\alpha$

$= V\cos\alpha \times \frac{2V\sin\alpha}{g}$

$= \frac{V^2 2\sin\alpha\cos\alpha}{g}$

$= \frac{V^2\sin 2\alpha}{g}$ ✓

Q.4 (a)(i)



$\frac{r}{2} = \frac{h}{3}$

$r = \frac{2h}{3}$

i) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h$

$= \frac{4\pi h^3}{27}$

i) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$0\cdot008 = \frac{4\pi h^2}{9} \times \frac{dh}{dt}$

$\frac{dh}{dt} = 0\cdot008 \times \frac{9}{4\pi h^2}$

$= \frac{0\cdot018}{\pi h^2}$

v) When $h = 1\cdot5$, $\frac{dh}{dt} = \frac{0\cdot018}{\pi \times 1\cdot5^2}$

$= 0\cdot0025$ m/s

) (i) $\frac{dT}{dt} = kAe^{kt}$

$= k(T-S)$

ii) $S = 28$, when $t=0$ $T = 100$

$\Rightarrow$ $100 = 28 + Ae^0$

$\Rightarrow$ $A = 72$

$T = 28 + 72e^{kt}$

When $t = 15$ $T = 80 \Rightarrow$

$80 = 28 + 72e^{15k}$

$\frac{52}{72} = e^{15k}$

$k = \frac{1}{15}\ln\frac{52}{72}$

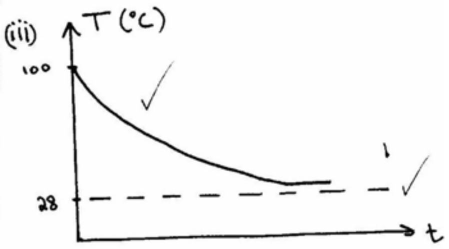
$= -0\cdot0216\ldots$

When $t = 30$,

$T = 28 + 72e^{30\times -0\cdot0216\ldots}$

$= 66°C$ (to nearest degree)

(iii)



5. (a) (i) $6!$ (ii) $5!$

) ${}^7P_3 = 210$

) $\frac{{}^nP_5}{{}^nC_3} = \frac{n!}{(n-5)!} \div \frac{n!}{(n-3)!\,3!}$

$= \frac{n!}{(n-5)!} \times \frac{(n-3)!\,3!}{n!}$

$= 3!\,(n-3)(n-4)$

$= 6(n^2 - 7n + 12)$

) $\frac{7!}{2!\,2!} = 1260$

) (i) ${}^9C_4 = 126$

(ii) ${}^7C_2 = 21$

(iii) ${}^7C_4 = 35$

(iv) ${}^9C_4 - {}^7C_4 = 91$ ((i) - (iii))

6

$$\begin{array}{r}
x^2 - 3x + 6 \\
x^2+1 \,\big)\, \overline{x^4 - 3x^3 + 7x^2 - 5x - 2} \\
\underline{x^4 \qquad\quad + x^2} \\
-3x^3 + 6x^2 - 5x \\
\underline{-3x^3 \qquad\quad - 3x} \\
6x^2 - 2x - 2 \\
\underline{6x^2 \qquad\quad + 6} \\
-2x - 8
\end{array}$$

$Q(x) = x^2 - 3x + 5$

$R(x) = -2x - 7$

(b) $P(-2) = 0 \Rightarrow -8a + 4b + 16 - 28 = 0$

ie $2a - b = -3$ ... ①

$P(1) = -27 \Rightarrow$ $a + b - 8 - 28 = -27$

ie. $a + b = 399$ ... ②

From ① + ② $a = 2$, $b = 7$

c) Let roots be $2\alpha, 3\alpha, 4\alpha$.

$2\alpha + 3\alpha + 4\alpha = -\frac{q}{p}$ ie. $9\alpha = -\frac{q}{p}$

$6\alpha^2 + 12\alpha^2 + 8\alpha^2 = \frac{r}{p}$ ie $26\alpha^2 = \frac{r}{p}$

$24\alpha^3 = -\frac{s}{p}$

$4qr = 4 \times -9p\alpha \times 26p\alpha^2$

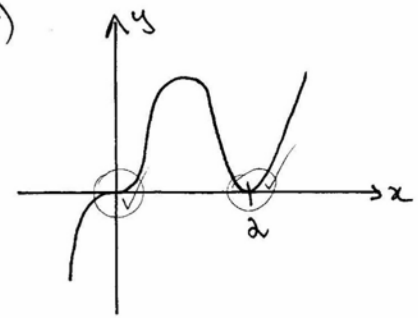
$= -936p^2\alpha^3$

$39ps = 39p \times -24p\alpha^3$

$= -936p^2\alpha^3$

$\therefore 4qr = 39ps$

d)



27 (a) (i) $-\sin x\, e^{\cos x}$ ✓

(ii) $\frac{\cos x}{\sin x}$ ✓ $(= \cot x)$

b) (i) $\frac{1}{x \ln x}$ ✓

(ii) $\int_e^{e^2} \frac{dx}{x \ln x} = \left[\ln(\ln x)\right]_e^{e^2}$ ✓

$= \ln(\ln e^2) - \ln(\ln e)$ ✓

$= \ln 2 - \ln 1$

$= \ln 2.$ ✓ $(= 0.69\ldots)$

c) $y = e^x \cos x$

$y' = -e^x \sin x + e^x \cos x$ ✓

$y'' = -e^x \cos x - e^x \sin x$
$\quad - e^x \sin x + e^x \cos x$

$= -2e^x \sin x$ ✓

$y'' - 2y' + ky = 0$

$-2e^x \sin x + 2e^x \sin x - 2e^x \cos x$
$\quad + k(e^x \cos x) = 0$

$ke^x \cos x - 2e^x \cos x = 0$

$k = 2.$ ✓

d) If $u = \tan x$

$du = \sec^2 x\, dx$

$= \frac{dx}{\cos^2 x}$ ✓

$I = \int \frac{\sin^3 x}{\cos^5 x}\, dx$

$= \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{dx}{\cos^2 x}$

$= \int u^3 \cdot du$ ✓

$= \frac{1}{4} u^4 + c$

$= \frac{1}{4} \tan^4 x + c$ ✓