KILLARA HIGH SCHOOL

YEAR 12 – 2008

HSC ASSESSMENT 2

MATHEMATICS EXTENSION 1

Time allowed: **2 hours**

Total marks: **84**

- **NOT ALL** questions are of equal value.
- Start each question in a new booklet.
- Show all necessary working.

Question 1 (Begin a new booklet) **Marks**

a) $(x-2)$ is a factor of the polynomial $P(x)=2x^3+x+a$. Find the value of a. 1

b) $\int_{-1}^{1}(1+5x)^4\,dx$ 2

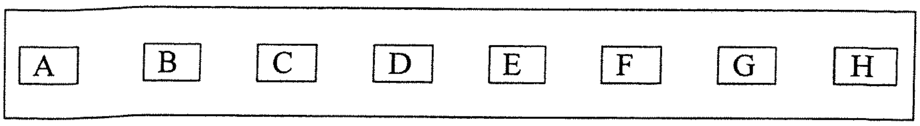
c) On a number plane, indicate the region defined by $y \geq |2x|+1$ and $y<3$. 3

d) Find the acute angle (to the nearest degree) between the line y - x + 1 = 0 and the curve $y=(x-1)^2$ at the point of intersection where x = 2. 3

e) Solve the inequality $\dfrac{2x+1}{x-1} \geq 3$. 3

Question 2 (Begin a new booklet) **Marks**

a)



A security lock has 8 buttons labeled as shown. Each person using the lock is given a 3-letter code.

i. How many different codes are possible if letters can be repeated and their order is important? 1

ii. How many different codes are possible if letters can not be repeated and their order is important? 1

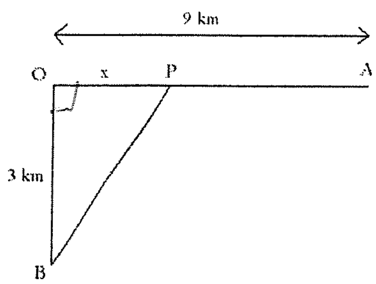
iii. Suppose that the lock operates by holding 3 buttons down together, given that the order is NOT important. How many different codes are possible? 1

Question 2 (continue)

a) Consider the equation $x^3 + 6x^2 - x - 30 = 0.$

One of the roots of this equation is equal to the sum of the other two roots. Find the values of the three roots. **4**

b)



The diagram shows a straight road OA, running due east. A motorist is in open country at B, 3 km due south of O. He must reach the point A as quickly as he can, where OA = 9 km. The car can travel at 80 km/h in open country and at 100 km/h along the road. He intends to proceed in a straight line to some point P on the road and then travel along the road.

i. Let x be the distance OP, show that the time taken to reach the point A is given by

$$T = \frac{\sqrt{x^2+9}}{80} + \frac{9-x}{100}$$ **2**

ii. Show that the minimum time for the entire journey is 6 ¾ min. **4**

Question 3 **(Begin a new booklet)** **Marks**

a) Prove the identity $\tan(45° + A) + \tan(45° - A) = \dfrac{2}{\cos 2A}$ **3**

b) The function $f(x)$ is defined by $f(x) = \dfrac{1}{x^2 + 1}$.

i. Show that $f(x)$ is an even function. **1**

ii. State the domain of $f(x)$. Find the intercepts of the curves $f(x)$ with the co-ordinate axes. **2**

iii. Write down the horizontal asymptotes. **1**

iv. Find and determine the nature of the turning point. **3**

v. Describe the behavior of $f(x)$ as $x \to \infty$ and $x \to -\infty$ **1**

vi. Draw a sketch of y = $f(x)$. **2**

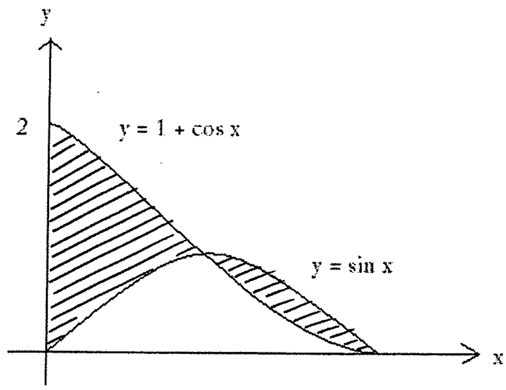
Question 4 **(Begin a new booklet)** **Marks**

a) Find $\int \frac{x}{\sqrt{1+x}}\,dx$. Using substitution $u = 1 + x$. **4**

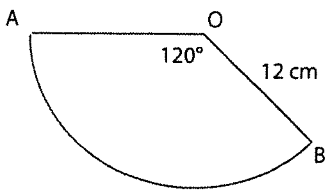
b) i. By **solving** for x, show that the solutions to the equation

$\cos x - \sin x = -1$ for $0 \le x \le \pi$ are $x = \frac{\pi}{2}$ *and* $x = \pi$. **3**

ii. Hence find the shaded area bounded by the graphs of $y = \sin x$ and $y = 1 + \cos x$ (shown below), from $x = 0$ *and* $x = \pi$. **3**



c)



A piece of paper is cut in the shape of a sector of a circle. The radius is 12 cm and the angle at the centre is 120°. The straight edges of the sector are placed together so that a cone is formed.

i. Show that the base of the cone has radius 4 cm. **1**

ii. Show that the cone has perpendicular height $8\sqrt{2}$ cm. **1**

iii. Hence find, the exact form, the volume of the cone. **1**

iv. Find the curved surface area of the cone. **1**

Question 5 **(Begin a new booklet)** **Marks**

a) Find $\lim_{x \to 0} \dfrac{\sin 6x}{5x}$. **2**

b) Find $\dfrac{dy}{dx}$ for the following:

i. $y = \tan\dfrac{x}{2} + \sin^3 x$ **2**

ii. $y = e^{-x} \sin x$ **2**

c) Find:

i. $\int \cos\left(2x - \dfrac{\pi}{3}\right) dx$ **1**

ii. $\int \sin^2\left(\dfrac{x}{2}\right) dx$ **2**

d) On the same axes, sketch the curves y = sin 2x, y = cos x and y = cos x + sin 2x, for $0 \leq x \leq \pi$. **4**

Question 6 **(Begin a new booklet)** **Marks**

a) Given that $\log_2 5 = 2.32$ and $\log_2 10 = 3.32$, find $\log_2 0.025$. **2**

b) Differentiate with respect to x

i. $\ln(3x^2 - 1)$ **1**

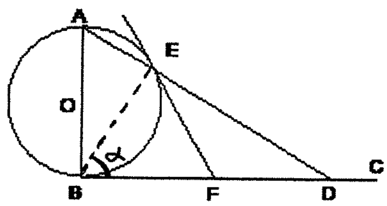
ii. $\dfrac{2x+1}{e^x}$ **2**

c) Find $\int \dfrac{2x-1}{x+3}\, dx$ **2**

d) Find the exact volume of the solid formed when the region bounded by the x-axis and the curve $y = e^x + e^{-x}$, between $x = -1 \, and \, x = 1$, is rotated around the x-axis. **4**

Question 7 **(Begin a new booklet)** **Marks**

a) Find the coordinates of the focus and the vertex of the parabola $x^2 = -12y + 24$. **2**



b) In the diagram, AB is a diameter of the circle, centre O, and BC is tangential to the circle at B. The line AED intersect the circle at E and BC at D. The tangent to the circle at E intersects BC at F. Let $\angle$ EBF = α radian.

i. Copy the diagram into your writing booklet.

ii. Prove that $\angle FED = \dfrac{\pi}{2} - \alpha$. **4**

iii. Prove that BF = FD. **2**

Q1

(i)

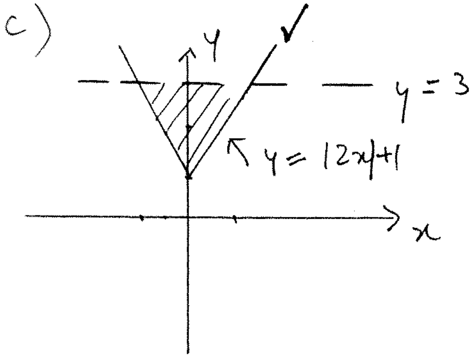
a) $P(2) = 2 \times 2^3 + 2 + a = 0$

$a = -18$ ✓

b) $\int_{-1}^{1} (1+5x)^4 \, dx = \left[\frac{1}{25}\left[1+5x\right]^5 \right]_{-1}^{1}$ ✓

$= \frac{1}{25}\left(6^5 - (-4)^5\right) = 352$ ✓

c)



✓ correct region

d) $y_1 = x - 1$, $m_1 = 1$

$y_2 = x^2 - 2x + 1$

$y_2' = 2x - 2$ at $x = 2$

$m_2 = 2$ ✓

$\tan\theta = \left| \frac{1-2}{1+1\times2} \right| = \frac{1}{3}$ ✓

$\theta \doteqdot \quad ^\circ$ ✓

e) $(2x+1)(x-1) \geq 3(x-1)^2$ ✓ $x \neq 1$

$3(x-1)^2 - (2x+1)(x-1) \leq 0$

$(x-1)(3x-3-2x-1) \leq 0$

$(x-1)(x-4) \leq 0$ ✓

$1 < x \leq 4$ is the solution ✓

12 marks

No half mark

Q2

(i) a) $8^3 = 512$ ✓

b) $8 \times 7 \times 6 = {}^8P_3$

$= 336$ ✓

c) ${}^8C_3 = 56$ ✓

(ii)

let the roots be α, β, γ

$\alpha = \beta + \gamma$

$2\beta + 2\gamma = -6$ ✓

$\beta + \gamma = -3 \Rightarrow \beta = -3 - \gamma$ ①

$\beta \cdot \gamma \cdot (\beta + \gamma) = +30$

$-3\beta\gamma = +30$ ✓

$\beta\gamma = -10 \leftarrow$ sub ① into

$\Rightarrow (-3-\gamma)\gamma = -10$

$\gamma^2 + 3\gamma - 10 = 0$

$(\gamma+5)(\gamma-2) = 0$ ✓

$\gamma = -5, 2$

$\alpha = -3$

$\therefore$ The 3 roots are : $2, -3$ & -5 ✓

(iii) a) $BP = \sqrt{x^2+9}$ $\quad PA = 9 - x$ ✓

The time taken to travel from :

$B \to P$ is $\dfrac{\sqrt{x^2+9}}{80}$ and

$P \to A$ is $\dfrac{9-x}{100}$ ✓

$\therefore$ Total time for the Journey :

$$T = \frac{\sqrt{x^2+9}}{80} + \frac{9-x}{100}$$

b) $$\frac{dT}{dx} = \frac{2x}{2 \times 80\sqrt{x^2+9}} - \frac{1}{100}$$

$$= \frac{x}{80\sqrt{x^2+9}} - \frac{1}{100} = 0$$ ✓

$\Rightarrow 10x - 8\sqrt{x^2+9} = 0$

$100x^2 = 64x^2 + 576$

$36x^2 = 576$ ✓

$x^2 = 16$ ✓

$x = 4$

x	3	4	5
$f'(x)$	↘	0	↗

$\therefore$ $x = 4$ is a min.

when $x = 4$

$$T = \frac{\sqrt{16+9}}{80} + \frac{9-4}{100} = 0.1125 \text{ h}$$

$= 6\frac{3}{4}$ min ✓

13 marks

$\boxed{Q3}$

(i) $\tan(45^\circ + A) + \tan(45^\circ - A) = \dfrac{2}{\cos 2A}$

$$L.H.S = \frac{1+\tan A}{1-\tan A} + \frac{1-\tan A}{1+\tan A}$$

$$= \frac{(1+\tan A)^2 + (1-\tan A)^2}{1-\tan^2 A} \quad ✓$$

$$= \frac{\dfrac{(\cos A + \sin A)^2}{\cancel{\cos^2 A}} + \dfrac{(\cos A - \sin A)^2}{\cancel{\cos^2 A}}}{\dfrac{\cos^2 A - \sin^2 A}{\cancel{\cos^2 A}}} \quad ✓$$

$$= \frac{\cos^2 A + 2\cancel{\cos A}\sin A + \sin^2 A + \cos^2 A - 2\cancel{\sin A}\cos A + \sin^2 A}{\cos 2A}$$

$$= \frac{2(\cos^2 A + \sin^2 A)}{\cos 2A} = \frac{2}{\cos 2A} = R.H.S \quad ✓$$

(ii)

a) $f(-x) = \dfrac{1}{(-x)^2+1} = \dfrac{1}{x^2+1} = f(x)$ ✓

b) when $x = 0$ $\quad f(x) = \dfrac{1}{0+1} = 1$ ✓

domain: all real x. ✓

c) $\lim\limits_{x\to\infty} \dfrac{1}{x^2+1} \quad \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}+1} = 1$

$y = 1$ is horizontal asymptote. ✓

d) $f'(x) = -2x(x^2+1)$

$$= \frac{-2x}{(x^2+1)^2} = 0 \quad ✓$$

$\Rightarrow \quad x = 0, \; y = 1$

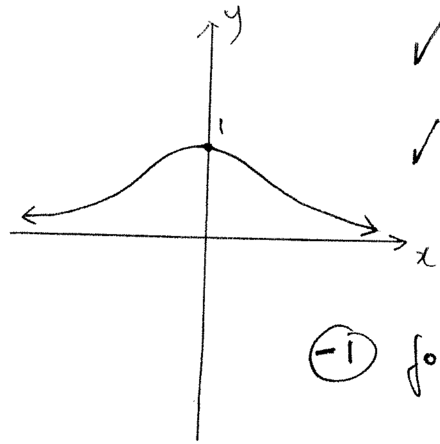
x	-1	0	1
$f'(x)$	/	0	\

✓

$\therefore$ $(0,1)$ is a min. point.

e) as $x \to \infty$, $f(x) \to 0$ $\quad , \quad$ $x \to -\infty$, $f(x) \to 0$ ✓

f)



✓ correct shape

✓ labelling

(-1) for any missing

$\boxed{12 \text{ marks}}$

$\boxed{Q4}$

(i) $\displaystyle\int \frac{x}{\sqrt{1+x}}\,dx$

$u = 1+x \Rightarrow x = u-1$

$du = dx$

$$\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du$$

$$= \int (u-1)u^{-\frac{1}{2}}\,du$$

$$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}$$

$$= \frac{2}{3}(x+1)\sqrt{x+1} - 2\sqrt{x+1} + C$$

(ii) $\cos x - \sin x = -1$

$r = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \frac{1}{1} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore \sqrt{2}\cos\left(x+\frac{\pi}{4}\right) = -1$

$\cos\left(x+\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$x+\frac{\pi}{4} = \frac{3\pi}{4}, \frac{5\pi}{4}$

b) $\displaystyle A = \int_0^{\pi/2} 1+\cos x - \sin x + \int_{\pi/2}^{\pi} \sin x - 1 - \cos x\,dx$

$$= \left[x + \sin x + \cos x\right]_0^{\pi/2} + \left[-\cos x - x - \sin x\right]_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2} + 1 - 1 + 1 - \pi - \left(-\frac{\pi}{2} - 1\right) = 2$$

(iii)

a) $l_{AB} = 12 \times \frac{120\times\pi}{180} = 8\pi$

$8\pi = 2\pi r \Rightarrow r = \frac{8\pi}{2\pi} = 4$ cm

b) $h = \sqrt{12^2 - 4^2}$

$= \sqrt{128}$

$= 8\sqrt{2}$ cm

c) $V = \frac{1}{3}\pi \times 4^2 \times 8\sqrt{2}$

$= \frac{128\pi\sqrt{2}}{3}\pi\,\text{cm}^3$

d) $A_{sector} = \frac{1}{2} \times 12^2 \times \frac{120\times\pi}{180} = 48\pi\,\text{cm}^2$

$\boxed{\text{14 marks}}$

Q5. [13 marks]

(i) $\lim_{x\to 0} \frac{\sin 6x}{5x} = \lim_{x\to 0} \frac{6}{5}\frac{\sin 6x}{6x}$ ✓

$= \frac{6}{5}$ ✓

(ii) a) $y' = \frac{1}{2}\sec^2\frac{x}{2} + 3\sin^2 x\cos x$ ✓ ✓

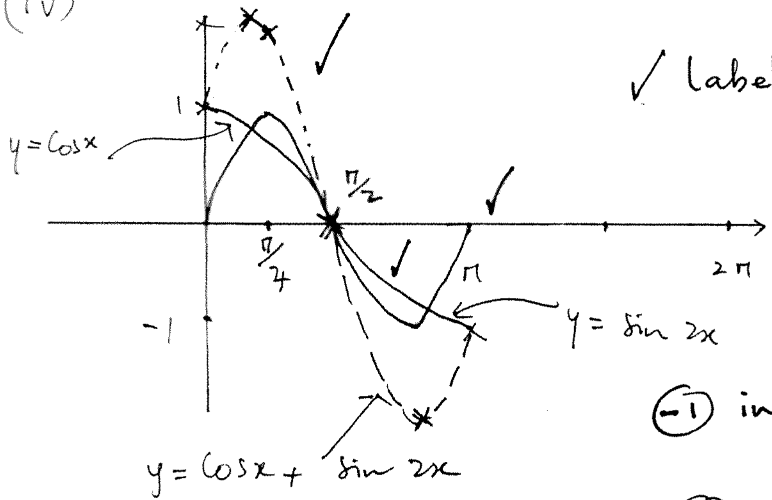
b) $y' = -e^{-x}\sin x + e^{-x}\cos x$ ✓ ✓

(ii) a) $\int \cos\left(2x - \frac{\pi}{3}\right) dx = \frac{1}{2}\sin\left(2x - \frac{\pi}{3}\right) + c$ ✓

b) $\int \sin^2\frac{x}{2}\, dx = \int \frac{1}{2} - \frac{1}{2}\cos x\, dx$ ✓

$= \frac{x}{2} - \frac{1}{2}\sin x + c$ ✓

(iv)



✓ labelling

(-1) incorrect scale/shape

Q6. [11 marks]

(i) $\log_2 0.025 = \log_2 \frac{5^2}{1000} = 2\log_2 5 - 3\log_2 10$

$= 2\times 2.32 - 3\times 3.32$ ✓

$= -5.32$

(ii) a) $\frac{d}{dx}\ln(3x^2-1) = \frac{6x}{3x^2-1}$ ✓

b) $\frac{2e^x - e^x(2x+1)}{e^{2x}}$ ✓

$= \frac{e^x(2-2x-1)}{e^{2x}} = \frac{1-2x}{e^x}$ ✓

(iii) $\int \frac{2x-1}{x+3}\, dx = \int \frac{2(x+3)}{x+3} - \frac{7}{x+3}\, dx$ ✓

$= 2x - 7\ln(x+3) + c$ ✓

(iv) $y^2 = (e^x + e^{-x})^2 = e^{2x} + 2 + e^{-2x}$ ✓

$V = \pi\int_{-1}^{1} e^{2x} + 2 + e^{-2x}\, dx$

$= \pi\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_{-1}^{1}$ ✓

$= \pi\left(\frac{e^2}{2} + 2 - \frac{e^{-2}}{2} - \frac{e^{-2}}{2} + 2 + \frac{e^2}{2}\right)$ ✓

✓

Q.1

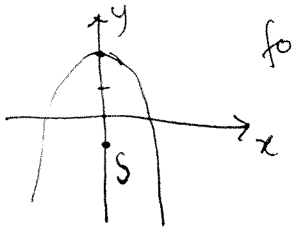
(i) $x^2 = -12(y-2)$

$= -4 \times 3 (y-2)$

Vertex : $(0,2)$ ✓

focus : $(0,-1)$ ✓

8 marks



(ii)

b) $BF = FE$ (tangents to the circle from a point) ✓

$\therefore \triangle BEF$ is isosceles. ✓

hence $\angle BEF = \alpha$

$\angle AEB = 90°$ (angle in a semi circle) ✓

$\therefore \angle BED = \pi - \pi/2 = \pi/2$

$\angle FED = \angle BED - \angle BEF$ ✓

$= \pi/2 - \alpha$

c)

$\angle EDB = \pi - (\pi/2 + \alpha)$ (angle sum of $\triangle$)

$= \pi/2 - \alpha$ ✓

$\therefore \angle EDB = \angle FED$ hence ✓

$\triangle FED$ is isosceles $\therefore BF = FD$.

(-1) no reason each time.