KILLARA HIGH SCHOOL

ASSESSMENT TASK 1
2009
Mathematics Extension 1

STUDENT NUMBER:______________________________

TIME ALLOWED: 70 minutes

INSTRUCTIONS: This paper consists of 4 questions
ANSWER all four questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

- applies appropriate techniques from the study of calculus and series to solve problems (H5)
- uses the derivative to determine the features of the graph of a function(H6)
- uses the relationship between functions and their derivatives(HE4)
- evaluates mathematical solutions to problems and communicates them in an appropriate form(HE7)
- solves problems involving permutations and combinations, and polynomials (PE3)

OUTCOMES	MARKS
1. Polynomials	/14
2. Permutations and Combinations	/10
3. Geometrical Applications of Differentiation	/10
4. Mathematical Induction	/ 6
	/40

This assessment task constitutes 10% of the Higher School Certificate Course Assessment.

QUESTION 1 (14 Marks) **Start a new booklet**

(a) For the polynomial $H(x) = 2x^3 - x^2 - 8x + 4$,

(i) find the remainder when $H(x)$ is divided by $x + 3$. **1**

(ii) Show that x-2 is a factor of $H(x)$. **1**

(iii) Find the intercepts that the graph of $y = H(x)$ makes with the x axis. **2**

(b) The polynomial $x^3 - 2x^2 + kx + 24 = 0$ has roots α, β and γ.

(i) Find the value of $\alpha + \beta + \gamma$ **1**

(ii) Find the value of $\alpha\beta\gamma$ **1**

(iii) It is known that two of the roots are equal in magnitude but opposite in sign. Find the third root and hence find the value of k. **2**

(c) The polynomial $P(x) = 2x^4 - ax^2 + bx - 1$ is exactly divisible by $x + 1$ and has a remainder of -3 on division by $(x - 2)$. Find the values of a and b. **3**

(d) Find the quotient, $Q(x)$, and the remainder, $R(x)$, when the polynomial $P(x) = x^4 - x^2 + 1$ is divided by $x^2 + 1$. **3**

QUESTION 2 (10 Marks) **Start a new booklet**

(a) The mathematics staff currently consists of 6 male and 7 female teachers.

(i) A team of 4 males and 5 females is to be chosen. **1**
Write an expression for the number of ways this can be done.

(ii) How many teams of 9 be chosen which contain at least 5 female teachers? **2**

(b) How many arrangements of the letters SCOOBIEDOO are possible? **2**

(c) Six students are to be seated in a row on the stage for an assembly.
How many ways can they be placed if:

(i) there are no restrictions on where they sit? **1**

(ii) two particular students insist on sitting next to each other? **1**

(iii) two particular students refuse to sit next to each other? **1**

(d) Solve ${}^{n}C_2 = 66$ **2**

QUESTION 3 (10 Marks) **Start a new booklet**

(a) The doctors report said 'Although the patient's temperature is still rising, attempts to bring it down to normal are having their effect.
The patient's temperature is C°.

(i) Jo said ' $\dfrac{dC}{dt} > 0$ '. Is Jo correct? Explain your answer. **1**

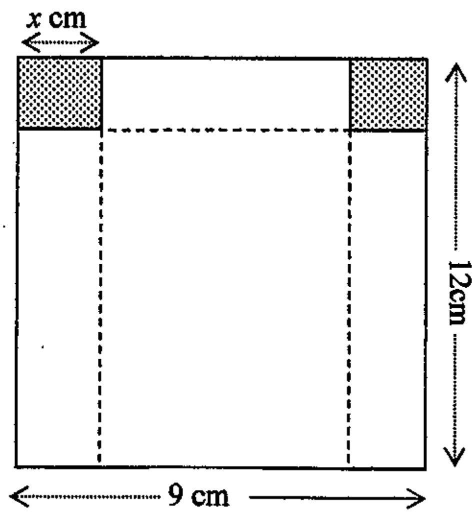
(ii) What conclusion can Jo draw about $\dfrac{d^2C}{dt^2}$? **1**
You must explain your answer.

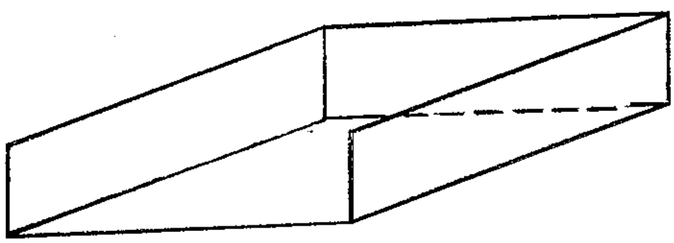
(b) A rectangle of cardboard measures 12cm by 9cm. From two corners, squares of side x cm are removed, as shown in the diagram below.
The remainder is folded along the dotted lines to form a tray.

(i) Show that the volume V cm³, of the tray is given by **1**

$$V = 2x^3 - 33x^2 + 108x$$

(ii) Find the maximum possible volume of the tray. **3**





(b) Consider the function $f(x) = \dfrac{x}{x+2}$

(i) Show that $f\,'(x) > 0$ for all x in the domain **2**

(ii) State the equation of the horizontal asymptote of $y = f(x)$ **1**

(iii) Without using any further calculus, sketch the graph of $y = f(x)$ **1**

QUESTION 4 (6 Marks) **Start a new booklet**

(a) Prove by mathematical induction that for a positive integer n: **4**

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}\ .$$

(b) If the following statement is true for S_k, show that it is also true for S_{k+1}. **2**

$$1\times 2^0+2\times 2^1+3\times 2^2+\ldots+k\times 2^{k-1}=1+(k-1)2^k,\quad \text{for all } k\geq 1.$$

END OF EXAMINATION

2009 Marking Scheme

1 a) (i) $H(x) = 2x^3 - x^2 - 8x + 4$
$H(-3) = 2(-3)^3 - (-3)^2 - 8(-3) + 4$
$H(-3) = -35$ — Award 1 for correct answer
$\therefore$ remainder is -35

(ii) If $x-2$ is a factor, $H(2) = 0$ — MUST HAVE
$H(2) = 2(2)^3 - 2^2 - 8(2) + 4$ — 1. If $x-2$ is factor, $H(2)=0$
$= 16 - 4 - 16 + 4$ — & full substitution
$= 0$ — [because its a show question].
$H(2) = 0$, $\therefore x-2$ is factor

(iii) $H(x) = (x-2)(2x^2 + 3x - 2)$ — Award 1 for correct answer.
$= (x-2)(2x-1)(x+2)$
Intercepts occur when $H(x) = 0$
$\therefore$ intercepts are $x = 2$, $x = \frac{1}{2}$, $x = -2$

b) $x^3 - 2x^2 + kx + 24$

(i) $\alpha + \beta + \gamma = \frac{-b}{a}$ — Award 1
$= 2$

(ii) $\alpha\beta\gamma = \frac{-d}{a}$ — Award 1
$= -24$

(iii) Let roots be $\alpha, -\alpha$ & β — Award 1 for correct value of 3rd root
$\therefore \alpha + -\alpha + \beta = 2$
$\beta = 2$

As 2 is a root, $2^3 - 2\times2^2 + k\times2 + 24 = 0$ — Award 1 for correct value of k
$8 - 8 + 2k + 24 = 0$
$k = -12$ — cfpa.

c) $P(x) = 2x^4 - ax^2 + bx - 1$

As $x+1$ is a factor, $P(-1) = 0$

$\therefore 2(-1)^4 - a(-1)^2 + b(-1) - 1 = 0$

$2 - a - b - 1 = 0$

$a + b = 1$ —— ① | Award 1

As -3 is remainder, $P(2) = -3$

$\therefore 2(2)^4 - a(2)^2 + b(2) - 1 = -3$

$32 - 4a + 2b - 1 = -3$

$-4a + 2b = -34$

$-2a + b = -17$ —— ② | Award 1

① − ② $3a = 18$

$a = 6$ —— (*)

(*) → ① $6 + b = 1$

$b = -5$ $\therefore a = 6, b = -5$ | Award 1 for correct sol'n.

(d)

$$\begin{array}{r}
x^2 - 2 \\
x^2+1 \overline{) x^4 - x^2 + 1} \\
x^4 + x^2 \\
\hline
-2x^2 + 1 \\
-2x^2 - 2 \\
\hline
3
\end{array}$$

$\therefore x^4 - x^2 + 1 = (x^2+1)(x^2-2) + 3$

$\therefore$ quotient $Q(x) = x^2 - 2$. | Award 1

remainder $R(x) = 3$

Question 2

a) (i) $^6C_4 \times {}^7C_5$

> Award 1 for correct EXPRESSION

(ii) At least 5 females means
5 females or 6 females or 7 females

ie $^7C_5 \times {}^6C_4 + {}^7C_6 \times {}^6C_3 + {}^7C_7 \times {}^6C_2$

> Award 1 for CNE

$= 315 + 140 + 15$

$= 470$

> Award 1 for correct answer

b) SCOOBIEDOO : Arrangements $= \dfrac{10!}{4!}$

> Award 2 for $\dfrac{10!}{4!}$

$= 151\,200$

> Award 1 for $10!$ or division by $4!$

c) (i) $6! = 720$

> Award 1

(ii) $5! \times 2 = 240$

> Award 1

(iii) $6! - 2 \times 5! = 480$

> Award 1

d) $^nC_2 = 66$ $\quad \dfrac{n!}{2!(n-2)!} = 66$ $\quad \therefore n > 2$

> Award 2 for $n = 12,\ n > 2$
> [must have $n > 2$

$\dfrac{n(n-1)}{2} = 66$

$n^2 - n - 132 = 0$

> Award 1 for $\dfrac{n!}{2!(n-2)!} = 66,\ n > 2$

$(n-12)(n+11) = 0$

$n = 12$ or -11

$n > 2, \quad \therefore n = 12$

Question 3

a) (i) Jo is correct. $\frac{dC}{dt} > 0$ means that as time increases, so does temperature

Award 1

(ii) $\frac{d^2C}{dt^2} < 0$ because as temperature is increasing, it is increasing at a decreasing rate

Award 1

b) (i) $V = (12 - x)(9 - 2x)x$

$= (108 - 33x + 2x^2)x$

$V = 2x^3 - 33x^2 + 108x$

Award 1
MUST HAVE ALL STEPS

(ii) MAX VOLUME when $\frac{dV}{dx} = 0$

$\frac{dV}{dx} = 6x^2 - 66x + 108$

$= 6(x^2 - 11x + 18)$

$= 6(x-9)(x-2) = 0$

$\frac{dV}{dx} = 0 \Rightarrow x = 9$ or $x = 2$

Award 1 for $x = 9$ or $x = 2$

$\frac{d^2V}{dx^2} = 12x - 66$

When $x = 9$, $\frac{d^2V}{dx^2} = 12 \times 9 - 66 > 0$ $\therefore$ minimum.

When $x = 2$, $\frac{d^2V}{dx^2} = 12 \times 2 - 66 < 0$

Award 1 for testing SP's
MUST SHOW SUBSTITIONS

$\therefore$ Maximum when $x = 2$

Maximum volume is $(12-2)(9-4)2$

$= 10 \times 5 \times 2$

$= 100 \text{ cm}^3$

Award 1 for correct answer

e) (i) $f(x) = \dfrac{x}{x+2}$

(i) $f'(x) = \dfrac{(x+2).1 - x.1}{(x+2)^2}$

$= \dfrac{2}{(x+2)^2}$

Award 1 for derivative.

As $(x+2)^2$ is always >0

$\dfrac{2}{(x+2)^2} > 0$

$\therefore \dfrac{dy}{dx} > 0 \therefore$ increasing

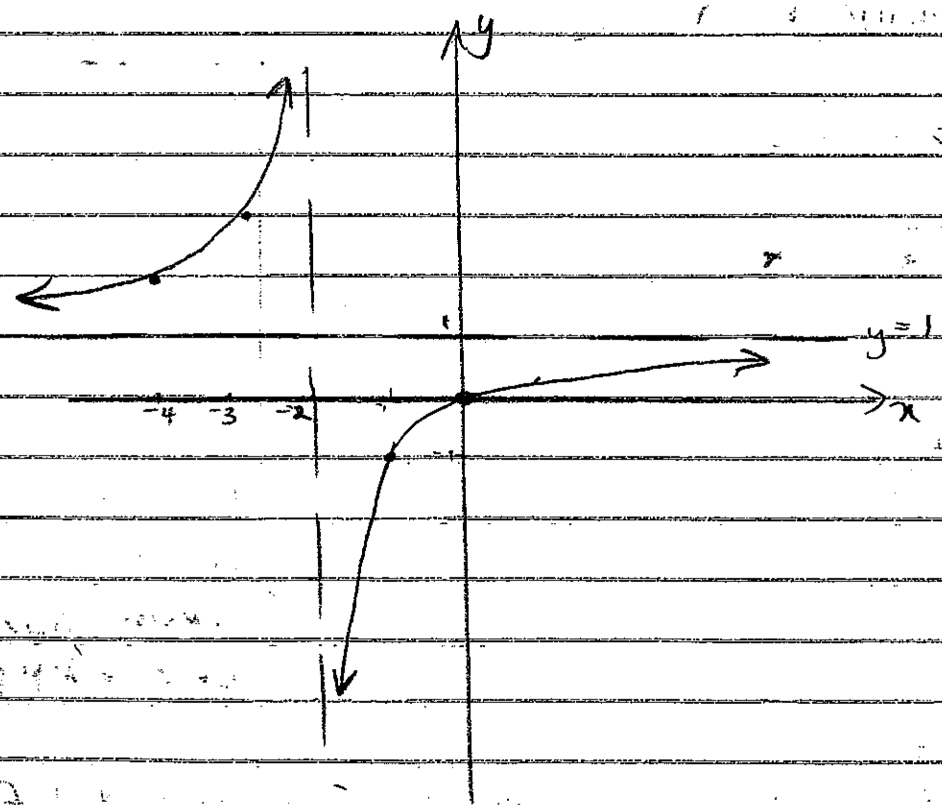
Award 1 for explanation AND $\dfrac{dy}{dx} > 0$

(ii) horizontal assymptote $y=1$

Award 1 MUST HAVE EQUATION

(iii) $x+2 \neq 0 \quad \therefore x \neq -2$

Award 1



MUST USE RULER & HAVE CONSISTENT SCALE

a) $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

1. Show result true for $n=1$

ie LHS $= \frac{1}{3}$ RHS $= \frac{1}{2\times1+1}$

$= \frac{1}{3}$

LHS = RHS

∴ result true for $n=1$.

Award 1. If student has written 'n=k' award 0

2. Assume result true for $n=k$.

ie $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

3. Show true for $n=k+1$

ie $\frac{1}{1\times3} + \frac{1}{3\times5} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k+1}{2(k+1)+1}$

Award 1 for correct substitution assumption statement into show statement

ie $\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$.

LHS $= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$

$= \frac{k+1}{2k+3}$

= RHS

Award 1 for correct resolution of LHS = RHS.

result true for $n=k+1$

As result is true for $n=1$, then result must be true for $n=1+1$, ie $n=2$, etc

Award 1 for correct statement

b) $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + k \times 2^{k-1} = 1 + (k-1)2^k$

show true for $n = k+1$

If $n = k+1$ then

$1 + 2^0 + 2 \times 2^1 + \ldots + k \times 2^{k-1} + (k+1)2^{k+1-1} = 1 + (k+1-1).2^{k+1}$

$= 1 + k.2^{k+1}$

ie -

$\underbrace{1 + (k-1)2^k + (k+1)2^k \qquad = 1 + k.2^{k+1}}_{\text{Award 1.}}$

LHS

$= 1 + k.2^k - 2^k + k.2^k + 2^k$

$= 1 + 2k.2^k$

$= 1 + k.2^1.2^k$

$= 1 + k.2^{k+1}$

$= RHS$

Award 1 for this step

MUST have EVERY STEP.

Watch for fudging