KILLARA HIGH SCHOOL

TERM 2 2009

TASK 3

EXTENSION 1 MATHEMATICS EXAMINATION
YEAR 12

Instructions:

Time allowed: 70 minutes

Approved calculators may be used.

Show ALL working clearly.

Marks may not be awarded for careless or badly arranged work.

Begin each question on a NEW page

Question 1 (15 marks) (Use a NEW page)

(a) (i) Find the largest possible domain of the function $f(x) = (x-2)^2 + 5$ for which $f(x)$ has an inverse function $g(x)$. **1**

(ii) Find $g(x)$ and hence sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes. **3**

(b). For the function $f(x) = 3\sin^{-1}(4x-1)$

(i) State the domain and range of $f(x)$. **2**

(ii) Hence sketch the graph of $f(x)$ showing the major features. **1**

(c) Differentiate $y = x\tan^{-1} x$ and hence evaluate the exact value of $\int_0^1 \tan^{-1} x\, dx$ **3**

(d) Evaluate:

(i) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}}$ **3**

(ii) Use the substitution $u = Sin\, x$, to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin^2 x}$ **2**

Question 2 on next page

Question 2 (10 marks) (Use a NEW page)

(a) A particle moves in a straight line with an acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2)$$

where x is the displacement in metres from an origin O after t seconds. Initially, the particle is 4 metres to the right of the origin O and has $v = -6 \text{ ms}^{-1}$.

(i) Show that $v^2 = 9(x-2)^2$ **2**

(ii) Find an expression for v and hence find x as a function of t. **2**

(iii) Explain whether the velocity of the particle is ever zero. **2**

(b) On a particular Saturday, the low-tide mark on a wharf was 1.4 m above sea bed and high tide 4 m above sea bed. The low-tide occurs at 9.05 a.m. and high-tide at 3.25 p.m. **4**

The top of the wharf is 4.3 m above seabed.

A barge tied to the wharf has its flat top 1.4 m above water level. Assuming that the rise and fall of the tide is a simple harmonic motion, calculate correct to the nearest minute, the time between low-tide and high-tide at which the top of the barge was exactly level with the top of the wharf.

(Draw a diagram to represent the information)

Question 3 on next page

Question 3 (9 marks) (Use a NEW page)

(a) The volume, $V\,\text{m}^3$, of usable wood in a tree of radius R metres can be modelled using the formula

$$\log_e V = 3\log_e(2R) - 0.81$$

(i) Using the approximation $e^{0.81} = 2.25$, show that the formula $\log_e V = 3\log_e(2R) - 0.81$ can be expressed as $V = \dfrac{32R^3}{9}$. **3**

(ii) The radius of the tree is increasing at a rate of 0.002 metre per year. At what rate is the usable volume of the wood in the tree increasing when the radius of the tree is 1.2 metre? **2**
(Answer in m^3/year to 2 significant years)

(b) An ice cube tray is filled with water at a temperature of $18°C$ and is placed in a freezer that has a constant temperature of $-19°C$.

The cooling rate is proportional to the difference between the temperature of the freezer and the temperature of the water, T.

That is, T satisfies the equations

$$\frac{dT}{dt} = -k\,(T+19) \text{ and } T = -19 + Ae^{-kt}$$

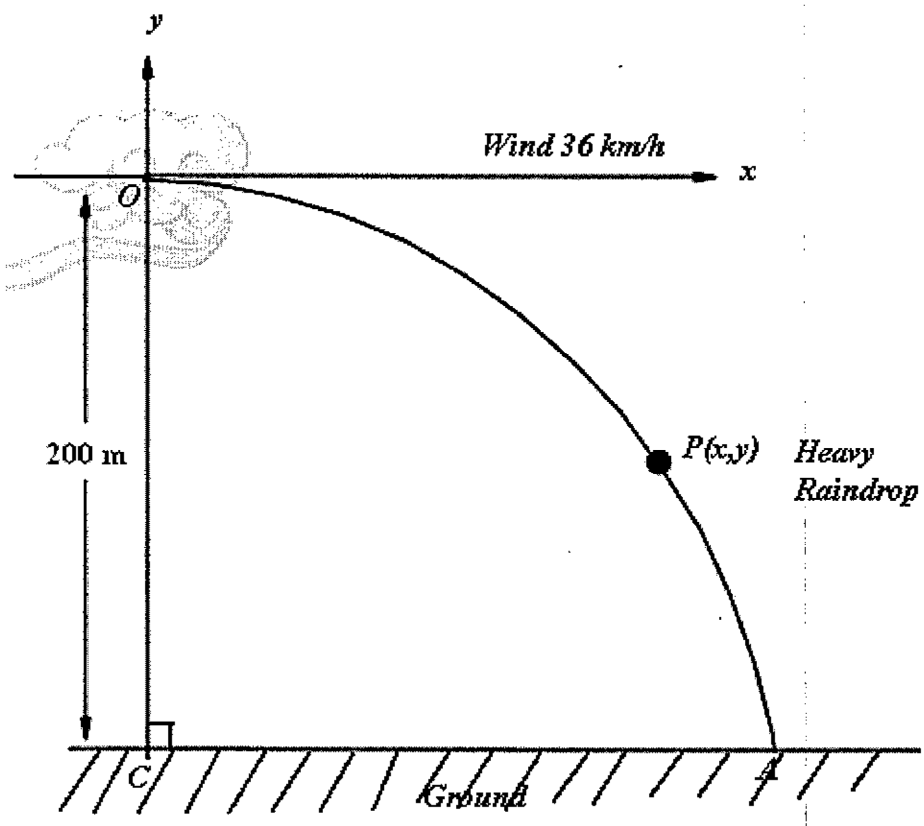
(i) Show that $A = 37$. **1**

(ii) After 5 minutes in the freezer the temperature of the water is $3°C$. Find the time for the water to reach $-18.9°C$. Answer to the nearest minute. **3**

Question 4 continues on next page

Question 4 (8 marks) (Use a NEW page)

A steady wind is blowing with a speed of 36km/h. From clouds moving horizontally with the wind, heavy raindrops fall to the ground 200 metres below.
(*Assume acceleration due to gravity* $g = 10ms^{-2}$)



(i) Derive the equations of motion. **2**

(ii) Find the time taken for the drop to reach the horizontal ground. **1**

(iii) Find the speed and angle at which a drop hits the horizontal ground. **3**

(iv) At what angle does a drop hit the ground when the wind speed is doubled? **2**

End of paper

2009 Ext 1 Task 3 Marking guide lines

(a) (i) $y = f(x) = (x-2)^2 + 5$

vertex is (2, 5)

$\therefore$ the domain of $f(x)$ is $x \geq 2$ for which $f(x)$ has an inverse function **(*1 mark*)**

(ii) $y = (x-2)^2 + 5$

$\therefore$ the inverse function is $x = (y-2)^2 + 5$

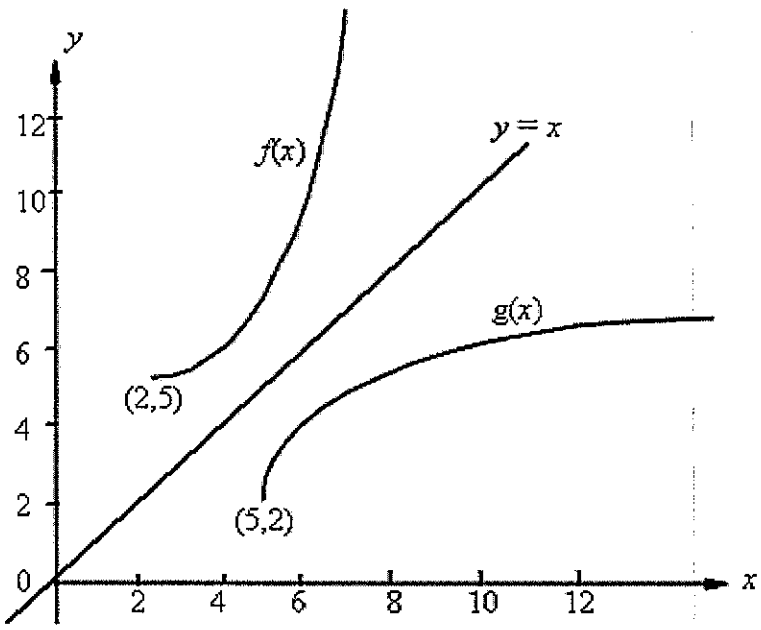
$$x - 5 = (y-2)^2$$

$$\therefore y - 2 = \pm\sqrt{x-5}$$

$$\therefore y = g(x) = 2 \pm \sqrt{x-5}$$ **(*1 mark*)**

The domain of $g(x)$ is $x \geq 5$ and range is $y \geq 2$.

$$y = g(x) = 2 + \sqrt{x-5}$$



(1 mark each for the correct graphs of y = f(x) and y = g(x)).

(b) (i) Domain: $-1 \leq 4x - 1 \leq 1$

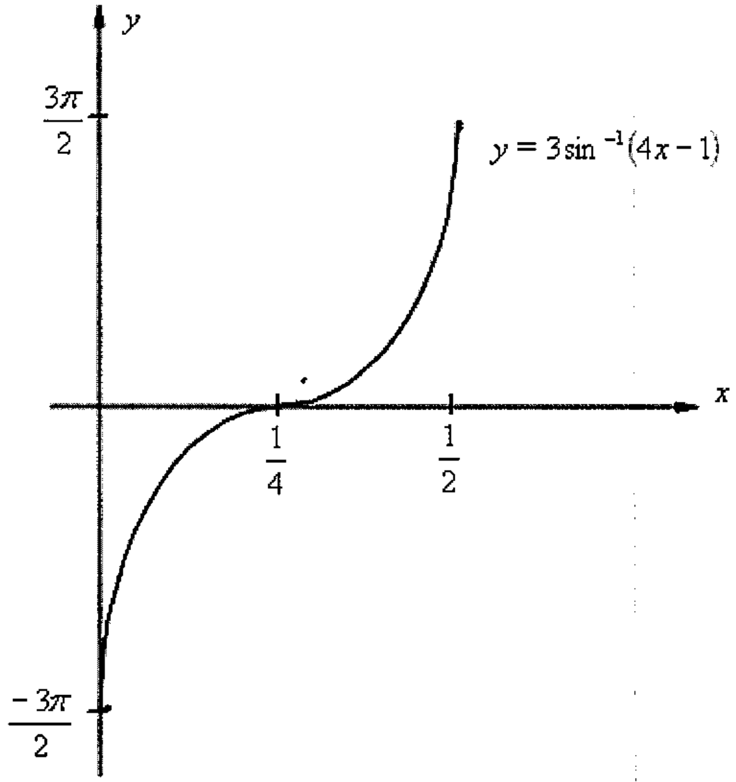
$$0 \leq 4x \leq 2$$

$$0 \leq x \leq \frac{1}{2}$$ **(*1 mark*)**

Range: $\frac{-\pi}{2} \leq \frac{y}{3} \leq \frac{\pi}{2}$

$$\frac{-3\pi}{2} \leq y \leq \frac{3\pi}{2}$$ **(*1 mark*)**

(ii)



(1 mark for the graph)

(c) $y = x\tan^{-1}x$

$$\frac{d(x\tan^{-1}x)}{dx} = x.\frac{1}{1+x^2} + \tan^{-1}x$$ ***(1 mark)***

$$\therefore \int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{x^2+1}dx$$ ***(1 mark)***

$$= \frac{\pi}{4} - \frac{1}{2}\left[\log_e\left(1+x^2\right)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$$ ***(1 mark)***

(d) (i) $$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{4\left(\frac{1}{4}-x^2\right)}} = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{\left(\frac{1}{4}-x^2\right)}}$$ **(1 mark)**

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x}{1/2}\right)\right]_0^{\frac{1}{2}}$$ **(1 mark)**

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$$ **(1 mark)**

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin^2 x}$

$$\left.\begin{array}{ll} \text{let } u = \sin x & \text{when } x = 0, \quad u = 0 \\ du = \cos x\, dx & \text{when } x = \frac{\pi}{2}, \quad u = 1 \end{array}\right\} \textbf{(1 mark)}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin^2 x} = \int_0^1 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^1 = \frac{\pi}{4} \qquad \textbf{(1 mark)}$$

Question 2

(a)(i) $\dfrac{d^2x}{dt^2} = 9(x-2)$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 9(x-2)$$

$$\therefore \frac{1}{2}v^2 = \int 9(x-2)dx$$

$$= 9\left(\frac{x^2}{2} - 2x\right) + C$$

When $t = 0$, $x = 4$ and $v = -6$.

$\therefore 18 = 9(8-8) + C \qquad \therefore C = 18 \qquad$ ***(1 mark)***

$$\therefore \frac{1}{2}v^2 = 9\left(\frac{x^2}{2} - 2x\right) + 18$$

$\therefore v^2 = 9x^2 - 36x + 36$

$= 9(x^2 - 4x + 4)$

$= 9(x-2)^2$ as required. ***(1 mark)***

(ii) $v^2 = 9(x-2)^2$

$V = \pm 3(x-2)$

$t = 0$, $x = 4$ and $v = -6$.

$\therefore v = -3(x-2) \qquad$ ***(1 mark)***

So, $\dfrac{dx}{dt} = -3(x-2)$

$$\therefore \frac{dt}{dx} = \frac{1}{-3(x-2)}$$

$$\therefore t = \frac{-1}{3}\int \frac{1}{x-2}dx$$

$$= \frac{-1}{3}\log_e(x-2) + C$$

$t = 0$, $x = 4$,

$$\therefore 0 = \frac{-1}{3}\log_e 2 + C$$

$\therefore\ C = \frac{1}{3}\log_e 2$

$\therefore\ t = \frac{-1}{3}\log_e(x-2) + \frac{1}{3}\log_e 2$

$= \frac{1}{3}\log_e \frac{2}{x-2}$

$3t = \log_e \frac{2}{x-2}$

$\frac{2}{x-2} = e^{3t}$

$x - 2 = 2e^{-3t}$

$x = 2(1 + e^{-3t})$ **(1 mark)**

(iii) $v = -3(x-2)$

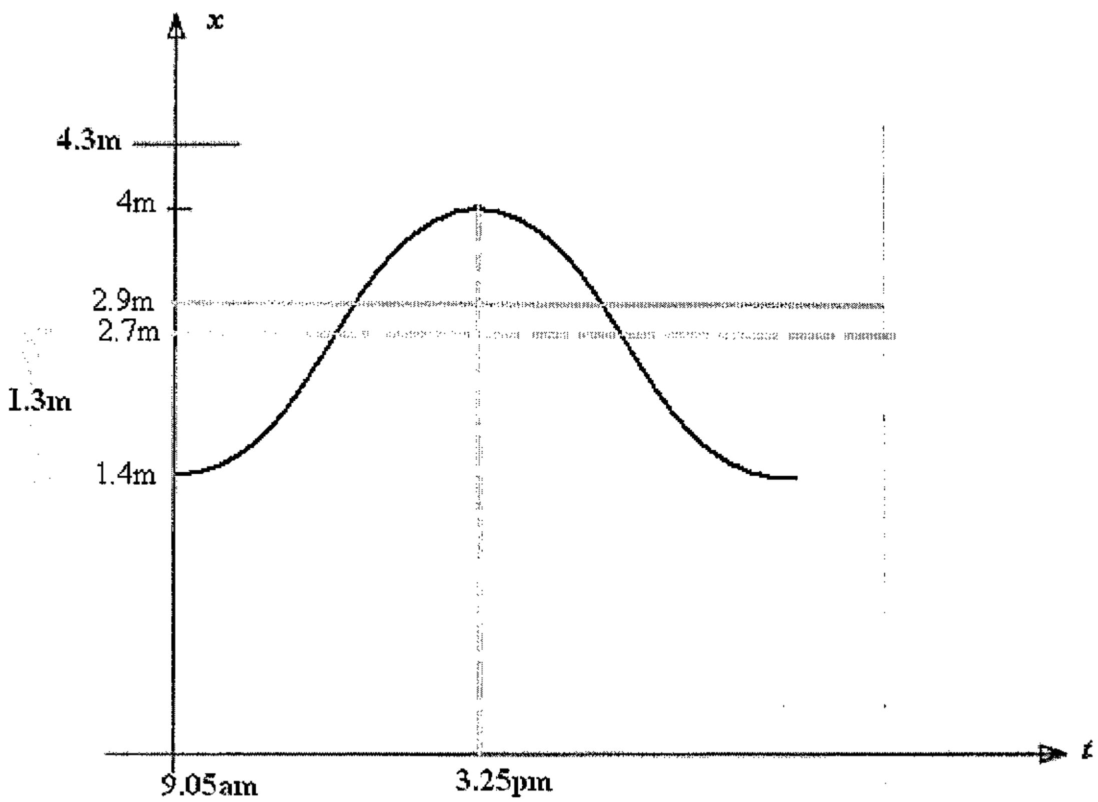
If $v = 0$, $x = 2$. **(1 mark)**

We have $x = 2(1 + e^{-3t})$

If $x = 2$, $2 = 2(1 + e^{-3t}) \rightarrow e^{-3t} = 0$ which has no solution.

$\therefore$ v is never zero. **(1 mark)**

(b)



(1 mark for reasonable diagram)

Half- period = 3.25 pm — 9.05am $= \frac{19}{3}$ hours

$\therefore$ period T = $\frac{38}{3}$ hours

$\therefore$ n = $\frac{2\pi}{T} = \frac{3\pi}{19}$

Displacement $x = 2.7 - 1.3 \cos\left(\frac{3\pi}{19}t\right)$ **(1 mark for correct equation)**

The barge is 1.4 m above water level.
$\therefore$ the barge is level with the wharf when the water level is 4.3 — 1.4 = 2.9m

To find the time when water level is 2.9m,

$$x = 2.7 - 1.3 \cos\left(\frac{3\pi}{19}t\right)$$

$$2.9 = 2.7 - 1.3 \cos\left(\frac{3\pi}{19}t\right)$$

$$0.2 = -1.3 \cos\left(\frac{3\pi}{19}t\right)$$

$$\text{Cos}\left(\frac{3\pi}{19}t\right) = \frac{-0.2}{1.3}$$

✓S | A
✓T | C

$$\left(\frac{3\pi}{19}t\right) = 1.725\ldots$$

t = 3.478…. **(Solves for t : 1 mark)**
= 3h 28 min 41 sec
= 3h 29 min

$\therefore$ time = 3h 29 min + 9.05 am = 12.34pm. **(1 mark correct time)**
(from diagram there is only one solution between 9.05am and 3.25pm.

Question 3

(a) (i) $\log_e V = 3\log_e(2R) - 0.81$

$$0.81 = 3\log_e(2R) - \log_e V$$

$$\log_e\left(\frac{(2R)^3}{V}\right) = 0.81 \quad \textbf{(1 mark)}$$

$$\frac{8R^3}{V} = e^{0.81}$$

$$\frac{8R^3}{V} = 2.25 \quad (\text{Given } e^{0.81} = 2.25) \quad \textbf{(1 mark)}$$

$$\frac{8R^3}{V} = \frac{9}{4}$$

$$V = \frac{4}{9} \times 8R^3$$

ie. $V = \frac{32R^3}{9}$ **(1 mark)**

(ii) $\frac{dR}{dt} = 0.002$ m^3/year

$V = \frac{32R^3}{9}$

$\therefore \frac{dV}{dR} = \frac{32}{9} \times 3R^2 = \frac{32R^2}{3}$ **(1 mark)**

$\frac{dV}{dt} = \frac{dV}{dR} \times \frac{dR}{dt}$

$= \frac{32R^2}{3} \times 0.002$

When $R = 1.2$, $\frac{dV}{dR} = \frac{32 \times (1.2)^2}{3} \times 0.002$ **(1 mark)**

$= 0.03072\ldots$

$= 0.031$m^3/year

(b) When t = 0, T = 18

$\therefore 18 = -19 + Ae^{-k \times 0}$

$\therefore A = 19 + 18 = 37$ **(1 mark)**

(ii) t = 5, T = 3

$3 = -19 + 37e^{-5k}$ **(1 mark)**

$e^{-5k} = \frac{22}{37}$

$k = \frac{-1}{5} \log_e \frac{22}{37}$ **(1 mark)**

$= 0.103975091$

When T = –18.9,

$-18.9 = -19 + 37 e^{-kt}$

$37 e^{-kt} = 0.1$

$e^{-kt} = \frac{0.1}{37}$

$-kt = \log_e\left(\frac{0.1}{37}\right)$

$\therefore t = \frac{\log_e\left(\frac{0.1}{37}\right)}{-k} = 56.8742\ldots$

$\therefore t = 57$ minutes. **(1 mark)**

Question 4

Now, 36km/h $= \dfrac{36000}{60 \times 60} = 10$m/s

Also, when t = 0, x = y = 0; $\dot{x}$ = 10m/s and $\dot{y}$ = 0 (clouds are moving horizontally)

(i)

Horizontally	Vertically
$\ddot{x} = 0$	$\ddot{y} = -g = -10\ \text{ms}^{-2}$
$\therefore \dot{x} = C_1$	$\dot{y} = \int -10\,dt$
$t = 0,\ \dot{x} = 10 \rightarrow C_1 = 10$	$= -10t + C_3$
$\therefore \dot{x} = 10$	$t = 0,\ \dot{y} = 0 \rightarrow C_3 = 0$
$x = \int 10\,dt = 10t + C_2$	$\dot{y} = -10t$
$t = 0, x = 0 \rightarrow C_2 = 0$	$y = -5t^2 + C_4$
$x = 10t$	$t = 0, y = 0 \rightarrow C_4 = 0$
	$y = -5t^2$

(1 mark each for correct derivation of equations of motion)

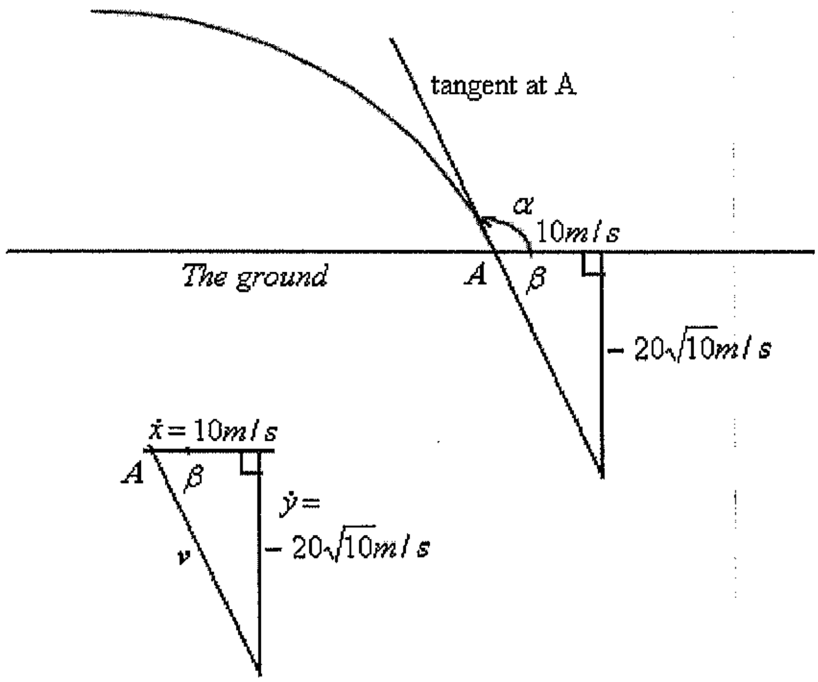
(ii) The raindrop falls 200m. ie. y = -200

$-5t^2 = -200$

$t^2 = 40$

$t = 2\sqrt{10}$ seconds ($t > 0$)

(iii)



(velocity Diagram 1 mark)

When the raindrop hits the ground (at A say), $t = 2\sqrt{10}$ seconds.

Then $\therefore \dot{x}$ = 10 m/s in the → direction and

$\dot{y} = -10t = -10 \times 2\sqrt{10} = -20\sqrt{10}\,m/s$ in the ↓ direction.

$|v| = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(10)^2 + (-20\sqrt{10})^2} = 10\sqrt{41}$ m/s **(1 mark)**

Also, $\tan\ \beta = \frac{|\dot{y}|}{\dot{x}} = \frac{\left|-20\sqrt{10}\right|}{10} = 2\sqrt{10}$

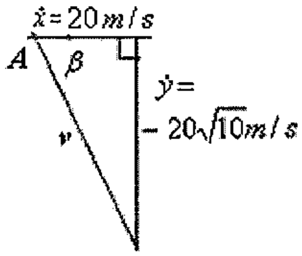
$\therefore\ \beta = 81°\ 1'$

$\alpha = 180 - \beta = 180 - 81°\ 1' = 98°\ 59'$

The raindrop hits the ground at an angle of 98° 59'. **(1 mark)**

(iv) When the wind speed doubles, $\dot{x} = 20$ m/s.
Using the initial conditions, the raindrop still falls 200m and therefore the time Taken to fall to the ground, still remains as $2\sqrt{10}$ seconds.

$\therefore\ \dot{y} = -10t = -10 \times 2\sqrt{10} = -20\sqrt{10}$ m/s **(1 mark)**



From the velocity diagram, $\tan\ \beta = \frac{|\dot{y}|}{\dot{x}} = \frac{\left|-20\sqrt{10}\right|}{20} = \sqrt{10}$

$\therefore\ \beta = 72°\ 27'$

$\alpha = 180 - \beta = 180 - 72°\ 27' = 107°\ 33'$

The raindrop hits the ground at an angle of 107° 33' . **(1 mark)**