# Killara High School

# Extension 1 Mathematics
# Term 1 2009

**General Instructions**
- Working Time – 2 hours
- Start each question in a new booklet
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

**Total Marks – 84**
- Attempt all questions
- All questions are of equal value

**Question 1** (Start a new booklet) **Marks**

i) Find the value of $x$ where $8^{2x-4} = 16^{x-1}.4^{-x}$ **2**

ii) Solve for $x$: $\frac{1}{2x-1} \leq 1.$ **2**

iii) The sum of the first n terms of a series is given by $S_n = \frac{n}{3}(n+1)(n+2)$

a. Show that the $n$th term is given by $T_n = n(n+1)$. **2**

b. Find the sum of the second 50 terms. **1**

iv) A geometric series has a limiting sum which has a value of $\frac{1}{1-w}$

and a common ratio of $\frac{1}{w}$. Find the first term. **1**

v) Solve for $x$, leave your answer in exact form

$$(x+\frac{2}{x})^2 - 6(x+\frac{2}{x}) - 7 = 0$$ **2**

vi) Evaluate $\lim\limits_{x \to 0} \frac{4\sin 2x}{3x}$ **2**

**Question 2** (Start a new booklet) **Marks**

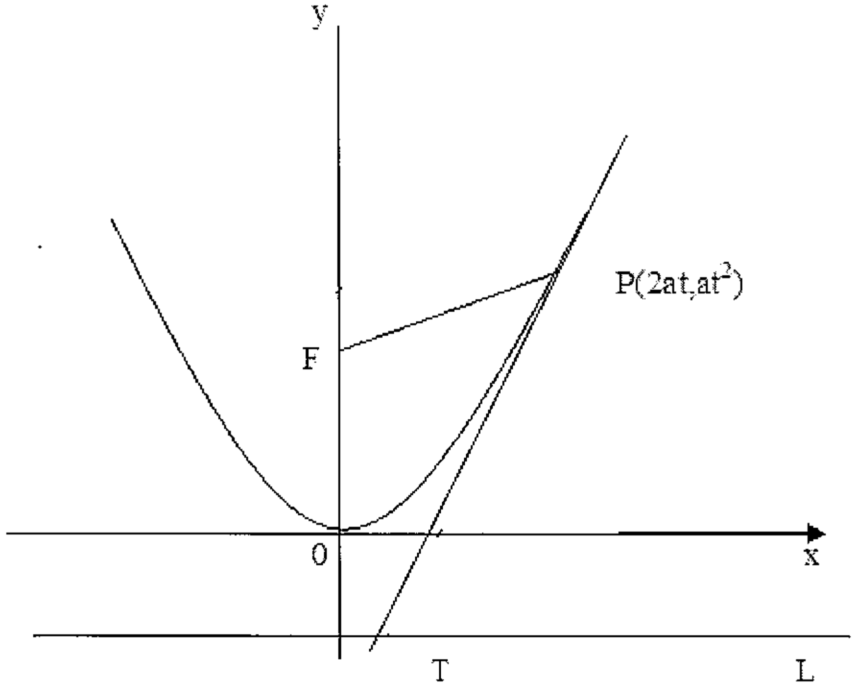
i) If $y=(x^2+2)^6$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ **2**

ii) a. Show that $(\sin A-\cos A)^2=1-\sin 2A$ **1**

b. Hence find the exact value of $(\sin 15^0-\cos 15^0)$. **1**

iii) A circle with area 25 $cm^2$ has an arc length of 2cm. Find the angle subtended at the centre of the circle by the arc. Answer in exact form. **2**

iv)




The tangent $tx-y-at^2=0$ at the point $P(2at,at^2)$ on the parabola $x^2=4ay$ cuts the directrix L at T. F is the focus of the parabola.

a. Find the coordinates of T . **1**

b. Show that TF is perpendicular to PF. **3**

v) Find the value of $\lim\limits_{x\to\infty}\frac{6(10^x)+9}{4(10^x)-3}$ **2**

**Question 3** (Start a new booklet) **Marks**

i) Evaluate each of the following indefinite integrals

a. $\int 6xe^{4x^2}\, dx$ **1**

b. $\int (2\cos 3x - \sin 2x)\ dx$ **2**

c. $\int \frac{1}{3}x(1+2x^2)^3\, dx$ **1**

d. $\int (\frac{3x^4}{2x^5-3})dx$ **1**

ii) Evaluate

a. $\int_0^1 \frac{e^x}{2e^x - 1}\, dx$ **2**

b. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx$ **2**

iii) Find the area bounded by the curve $y = 2\sin\frac{\pi}{2}x$, the $x$-axis

and the ordinates $x = 1$ and $x = 3$. **3**

**Question 4** (Start a new booklet)

i) Differentiate with respect to x:

a. $\sin(3x^2 - 2)$ **1**

b. $e^{3x^2-8x} + 7x^2$ **1**

c. $2xe^{6x}$ **2**

d. $\cot x$ **2**

e. $\ln\left(\dfrac{3x-4}{x+5}\right)$ **2**

ii) Six people enter an empty mini bus in which there are 14 seats; 7 on each side. In how many ways can they take their places if;

a. Anyone can occupy any seat. **1**

b. Jacquie sits in one of the four corner seats. **1**

iii) Differentiate $\sin^2(3x - \frac{\pi}{2})$ and find its value where $x = \frac{\pi}{4}$. **2**

**Question 5** (Start a new booklet) **Marks**

i) a. Show that $\dfrac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$ **2**

b. Hence find the value of $\tan^2\left(22\dfrac{1}{2}\right)^0$. **2**

ii) a. Express $\cos x - \sqrt{3}\sin x$ in the form $A\cos(x+\theta)$ **2**

b. Hence sketch the graph $y = \cos x - \sqrt{3}\sin x$ for $0 \le x \le 2\pi$ **4**

iii) Solve $2\cos^2\theta + 5\sin\theta = -1$ where $0 \le \theta \le 2\pi$ **2**

**Question 6** (Start a new booklet)

i) Use the process of mathematical induction to prove that

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\ldots\ldots+\frac{1}{x^n}=\frac{1}{x-1}-\frac{1}{x^n(x-1)} \quad \text{for } n\geq 1.$$ **5**

ii) a. Prove $2\cos^2 x = 1+\cos 2x$ **1**

b. If $y = 2\cos^2 2x - 1$, then evaluate the area between the curve the $x-$ axis and the ordinates $x=\frac{\pi}{16}$ and $x=\frac{3\pi}{8}$. **3**

iii) Calculate the volume generated (in simplest exact form), by rotating about the x-axis the curve $y=\frac{1}{\sqrt{4-x}}$ between $x=0$ and $x=3\frac{3}{4}$. **3**

**Question 7** (Start a new booklet) **Marks**

i) A polynomial P(x) has the following features. P(x) is monic and of degree 2. P(3) = P(-5). P(x) has a remainder – 6 when divided by x.

Find P(x). **3**

ii) By using the substitution u = $x^2$, evaluate $\int x\cos x^2 \; dx$ **3**

iii) Seven adults (including Joan and Mary) are to be selected at random from a group of ten adults.

a. How many unordered selections are possible if no restrictions exist? **1**

b. How many unordered selections are possible if Joan and Mary are not in the same group? **2**

iv) The area between the curve y = $\frac{3}{x+4}$, the y-axis and the ordinates x = $-3\frac{3}{4}$ and x = $-3\frac{1}{2}$ is rotated about the y-axis. Find the volume generated in exact form. **3**

**End of Paper**

## Question 1

i) $8^{2x-4} = 16^{x-1} \cdot 4^{-x}$

$2^{6x-12} = 2^{2x-4}$ ✓

$6x - 12 = 2x - 4$

$x = 2$ ✓

ii) $\frac{1}{2x-1} \leq 1 \quad (x \neq \frac{1}{2})$

$\left(\frac{1}{2x-1}\right)(2x-1)^2 \leq 1 \times (2x-1)^2$

$4x^2 - 4x + 1 \geq 2x - 1$

$2x^2 - 3x + 1 \geq 0$ ✓

$(2x-1)(x-1) \geq 0$

$x < \frac{1}{2}$ and $x \geq 1$ ✓

iii) a $S_n = \frac{n}{3}(n+1)(n+2)$

$S_1 = 2,\ S_2 = 8,\ S_3 = 20,\ S_4 = 40$

$\therefore$ Series is

$2,\ 6,\ 12,\ 20$ ✓

$1\times2,\ 2\times3,\ 3\times4,\ 4\times5, \ldots$

$T_1 \quad T_2 \quad T_3 \quad T_4$ ✓

$\therefore T_n = n(n+1)$

b Sum 2nd 50 $= S_{100} - S_{50}$

$= \left(\frac{100}{3}\right)(101)(102) - \left(\frac{50}{3}\right)(51)(52)$

✓ $= 299200$

iv

$S_\infty = \frac{a}{1-r}$

$\frac{1}{1-\omega} = \frac{a}{1-\frac{1}{\omega}}$

$\frac{1-\frac{1}{\omega}}{1-\omega} = a$

$a = \frac{\omega - 1}{-\omega^2 + \omega}$

$= -\frac{1}{\omega}$ ✓

v Let $A = x + \frac{2}{x}$

$A^2 - 6A - 7 = 0$

$A = 7$ or $A = -1$

$x + \frac{2}{x} = 7$ or $x + \frac{2}{x} = -1$ ✓

no solution

$x = \frac{7 \pm \sqrt{41}}{2}$ ✓

vi $\lim_{x \to 0} \frac{4 \sin 2x}{3x}$

$= \frac{4}{3} \times 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$ ✓

$= \frac{8}{3} \times 1$

$= \frac{8}{3}$ ✓

## Question 2

i) $y = (x^2+2)^6$

$y' = 12x(x^2+2)^5$ ✓

$y'' = 12(x^2+2)^5 + 120x^2(x^2+2)^4$ ✓

ii) a

$(\sin A - \cos A)^2$

$= \sin^2 A + \cos^2 A - 2\sin A\cos A$ ✓

$= 1 - \sin 2A$

b $(\sin 15 - \cos 15)$

$= \sqrt{1 - \sin(2\times 15)}$

$= \sqrt{1 - \sin 30}$

$= \frac{1}{\sqrt{2}}$ ✓

iii) $\pi r^2 = 25$

$r = \frac{5}{\sqrt{\pi}}$ ✓

$L = r\theta$

$2 = \frac{5}{\sqrt{\pi}}\theta$

$\theta = \frac{2\sqrt{\pi}}{5}^c$

$\therefore$ the angle at the centre is $\frac{2\sqrt{\pi}}{5}^c$. ✓

iv) a $tx - y - at^2 = 0$ — ①

$y = -a$ — ② The directrix

sub ② into ①

$tx = at^2 - a$

$x = \frac{at^2 - a}{t}$

$\therefore T = \left(\frac{at^2-a}{t}, -a\right)$ ✓

iv) b. $F = (0, a)$

$m_{PF} = \frac{at^2 - a}{2at} = \frac{t^2-1}{2t}$ ✓

$m_{TF} = \frac{-a-a}{\frac{at^2-a}{t}}$

$= \frac{-2at}{a(t^2-1)}$

$= \frac{-2t}{t^2-1}$ ✓

$m_{PF} \times m_{TF} = \frac{t^2-1}{2t} \times \frac{-2t}{t^2-1}$

$= -1.$

Hence $TF \perp PF$. ✓

v) $\lim_{n\to\infty} \frac{6(10^n)+9}{4(10^n)-3}$

$= \lim_{n\to\infty} \frac{6 + \frac{9}{10^n}}{4 - \frac{3}{10^n}}$ ✓

$= \frac{6+0}{4-0}$

$= 1\frac{1}{2}$ ✓

# Question 3

i) a $\int 6x e^{4x^2}\,dx$

$= \frac{3}{4}e^{4x^2} + c$ ✓

b $\int (2\cos 3x - \sin 2x)\,dx$

$= \frac{2}{3}\sin 3x + \frac{1}{2}\cos 2x + c$
✓ ✓

c $\frac{1}{3}\int 4x(1+2x^2)^3\,dx$

$= \frac{1}{48}(1+2x^2)^4 + c$ ✓

d $\int \frac{3x^4}{2x^5-3}\,dx$

$= \frac{3}{10}\ln(2x^5-3) + c$ ✓

ii) a $\int_0^1 \frac{e^x}{2e^x-1}\,dx$

$= \frac{1}{2}\int_0^1 \frac{2e^x}{2e^x-1}\,dx$

$= \frac{1}{2}\left[\ln(2e^x-1)\right]_0^1$ ✓

$= \frac{1}{2}\left[\ln(2e-1)\right]$ ✓

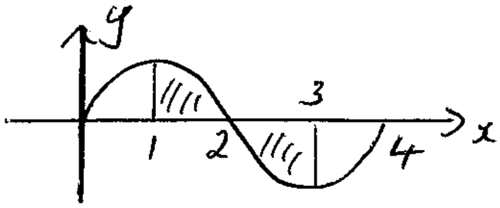
ii) b $\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx$

$= \left[\frac{1}{3}\sin^3 x\right]_0^{\pi/2}$ ✓

$= \frac{1}{3}\left[\sin^3\frac{\pi}{2} - \sin^3 0\right]$

$= \frac{1}{3}$ ✓

iii)




✓

Period = 4.

$\cancel{A = \int_1^3 2\sin\frac{\pi}{2}x\,dx}$

$\cancel{= \left[-\frac{2}{\pi}\cos\frac{\pi}{2}\right]_1^3}$

$\cancel{=}$

$A = 2\int_1^2 2\sin\frac{\pi}{2}x\,dx$ ✓

$= 4\cancel{\times}\left[-\cos\frac{\pi}{2}x\right]_1^2$

$= 4\left[-\frac{2}{\pi}\left(\cos\frac{2\pi}{2} - \cos\frac{\pi}{2}\right)\right]$

$= \frac{8}{\pi}$ units$^2$ ✓

Question 4

i) a $\frac{d}{dx}\sin(3x^2-2)$

$= 6x\cos(3x^2-2)$ ✓

b $\frac{d}{dx}\left(e^{3x^2-8x}+7x^2\right)$

$= (6x-8)\left(e^{3x^2-8x}\right)+14x$ ✓

c $\frac{d}{dx}\left(2xe^{6x}\right)$

$= (2x)(6e^{6x})+(e^{6x})(2)$ ✓

$= 2e^{6x}(6x+1)$ ✓

d $\frac{d}{dx}(\cot x)$

$= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ ✓

$= \frac{\sin x(-\sin x)-(\cos x)(\cos x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$ ✓

e $\frac{d}{dx}\left[\ln\left(\frac{3x-4}{x+5}\right)\right]$

$= \frac{d}{dx}\left[\ln(3x-4)-\ln(x+5)\right]$ ✓

$= \frac{3}{3x-4}-\frac{1}{x+5}$ ✓

ii)

a $^{14}P_6 = 2,162,160$ ✓

b. $(13\times12\times11\times10\times9)\times4$ ✓

$= 617,760$

iii) let $y=\sin^2\left(3x-\frac{\pi}{2}\right)$

$y' = 2\sin\left(3x-\frac{\pi}{2}\right)\times\cos\left(3x-\frac{\pi}{2}\right)\times3$

✓ $= 6\sin\left(3x-\frac{\pi}{2}\right)\cos\left(3x-\frac{\pi}{2}\right)$

when $x=\frac{\pi}{4}$.

$y' = 6\sin\frac{\pi}{4}\cos\frac{\pi}{4}$

$= 6\times\frac{1}{2}$

✓ $= 3$

# Question 5

i) a $LHS = \frac{1-\cos 2x}{1+\cos 2x}$

$= \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x - 1)}$ ✓

$= \frac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x$ ✓

$= RHS$

b $\tan^2 22\frac{1}{2} = \frac{1-\cos 45}{1+\cos 45}$ ✓

$= \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$

$= \frac{\sqrt{2}-1}{\sqrt{2}+1}$ ✓

ii) $A\cos(x+\theta)$

$= A\cos x \cos\theta - A\sin x \sin\theta$

$= 1\cos x - \sqrt{3}\sin x$

$\therefore A\cos\theta = 1$ ✓, $A\sin\theta = \sqrt{3}$

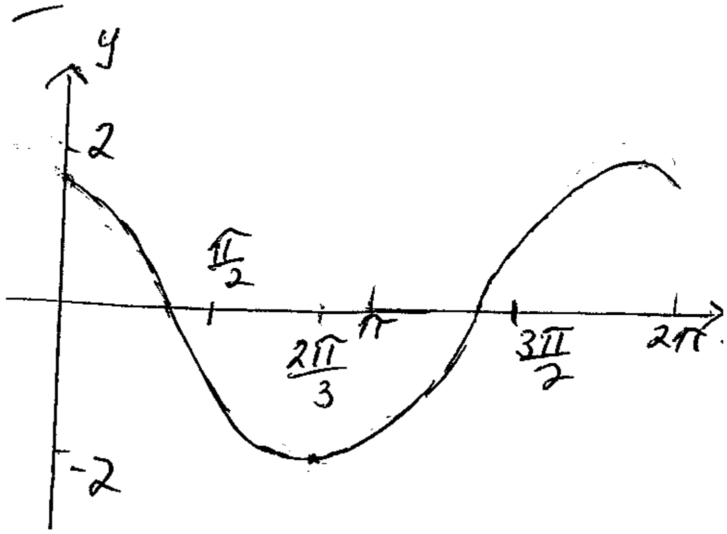
$\therefore \tan\theta = \sqrt{3}$

2, $\sqrt{3}$, 1, $\theta$

$\therefore A = 2$, $\theta = \frac{\pi}{3}$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos\left(x+\frac{\pi}{3}\right)$ ✓

ii b




$y = \cos x - \sqrt{3}\sin x$

$= 2\cos\left(x+\frac{\pi}{3}\right)$

Amplitude = 2 } ✓
Period = $2\pi$ }
labelling ✓
shape ✓
accuracy ✓

iii) $2\cos^2\theta + 5\sin\theta = -1$.

$2\sin^2\theta - 5\sin\theta - 3 = 0$

$(2\sin\theta + 1)(\sin\theta - 3) = 0$ ✓

$2\sin\theta + 1 = 0$, $\sin\theta - 3 = 0$

$\sin\theta = -\frac{1}{2}$ no solution

2, $\frac{\pi}{3}$, $\sqrt{3}$, 1

| S | A |
|---|---|
| T* | C* |

$\theta = \pi + \frac{\pi}{3}$, $2\pi - \frac{\pi}{3}$

$= \frac{4\pi}{3}$, $\frac{5\pi}{3}$ ✓

Question 6.

i. For $n=1$

$LHS = \frac{1}{x}$, $RHS = \frac{1}{x-1} - \frac{1}{x(x-1)} = \frac{x-1}{x(x-1)} = \frac{1}{x} = LHS$ ✓

$\therefore$ true for $n=1$.

Assume true for $n=k$.

ie $\frac{1}{x} + \frac{1}{x^2} + \dots + \frac{1}{x^k} = \frac{1}{x-1} - \frac{1}{x^k(x-1)}$ — ① ✓

Required to show true for $n=k+1$

ie $\frac{1}{x} + \frac{1}{x^2} + \dots + \frac{1}{x^k} + \frac{1}{x^{k+1}} = \frac{1}{x-1} - \frac{1}{x^{k+1}(x-1)}$

For $n=k+1$ ✓

$$LHS = \frac{1}{x} + \frac{1}{x^2} + \dots + \frac{1}{x^k} + \frac{1}{x^{k+1}}$$

$$= \frac{1}{x-1} - \frac{1}{x^k(x-1)} + \frac{1}{x^{k+1}} \quad \text{from ①}$$

$$= \frac{1}{x-1} - \frac{1}{x^k(x-1)} + \frac{1}{x^k \cdot x}$$

$$= \frac{1}{x-1} + \frac{-x+x-1}{x^k \cdot x \cdot (x-1)}$$

$$= \frac{1}{x-1} - \frac{1}{x^{k+1}(x-1)} \quad ✓$$

$= RHS$.

$\therefore$ If true for $n=k$ then it is true for $n=k+1$.

true for $n=1$, $\therefore$ true for $n=1+1=2$

true for $n=2$, $\therefore$ true for $n=2+1=3$ ✓

and so on.

$\therefore$ true for all $n \geq 1$.

## Question 6 (continued)

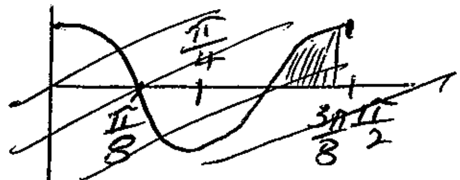
ii) a

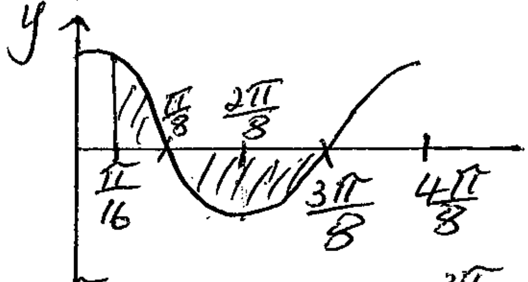
$$RHS = 1 + \cos 2x$$
$$= 1 + 2\cos^2 x - 1$$
$$= 2\cos^2 x$$
$$= LHS \quad ✓$$

b

$$y = 2\cos^2 2x - 1$$
$$y = \cos 4x$$

Period $= \frac{\pi}{2}$.




✓

$$A = \int_{\frac{\pi}{16}}^{\frac{\pi}{8}} \cos 4x \, dx + \left| \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \cos 4x \, dx \right| \quad ✓$$

$$= \left[\frac{1}{4}\sin 4x\right]_{\frac{\pi}{16}}^{\frac{\pi}{8}} + \left| \left[\frac{1}{4}\sin 4x\right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \right|$$

$$= \left(\frac{1}{4} - \frac{\sqrt{2}}{8}\right) + \left| \left(-\frac{1}{4} - \frac{1}{4}\right) \right|$$

$$= \left(\frac{3}{4} - \frac{\sqrt{2}}{8}\right) \text{ units}^2 \quad ✓$$

iii)

$$y = \frac{1}{\sqrt{4-x}}$$

$$y^2 = \frac{1}{4-x} \quad ✓$$

$$V = \int_0^{\frac{15}{4}} \pi y^2 \, dx$$

$$= -\pi \int_0^{\frac{15}{4}} \frac{-1}{4-x} \, dx \quad ✓$$

$$= -\pi \left[\ln(4-x)\right]_0^{\frac{15}{4}}$$

$$= -\pi \left[\ln \frac{1}{4} - \ln 4\right]$$

$$= 4\pi \ln 2 \text{ units}^3 \quad ✓$$

# Question 7

i) $P(x) = ax^2 + bx + c \quad a = 1$

$= x^2 + bx + c$

$P(3) = 9 + 3b + c$

$P(-5) = 25 - 5b + c$

$P(3) = P(-5)$

$\therefore 9 + 3b + c = 25 - 5b + c$ ✓

$b = 2$

$\therefore P(x) = x^2 + 2x + c$

$P(0) = -6$ ✓

$\therefore \quad c = -6$

$\therefore P(x) = x^2 + 2x - 6.$ ✓

ii) $u = x^2 \quad \therefore du = 2x\,dx.$ ✓

$\therefore \int x \cos x^2 \, dx$

$= \frac{1}{2}\int 2x \cos x^2 \, dx$

$= \frac{1}{2}\int \cos u \, du$ ✓

$= \frac{1}{2}(-\sin u) + c$

$= -\frac{1}{2}\sin x^2 + c$ ✓

iii)

a.

✓ ${}^{10}C_7 = 120$

b. There are 3 possiblities

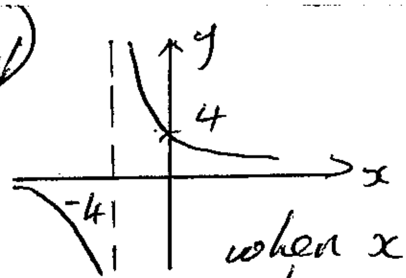
✓ Joan in Mary not
Mary in Joan not
Joan and Mary not in.

Total Number of Selections

$= {}^8C_6 + {}^8C_6 + {}^8C_7$ ✓

$= 64$ ✓

iv)




when $x = -3\frac{3}{4}, y = 12$
when $x = -3\frac{1}{2}, y = 6$

$x + 4 = \frac{3}{y}$

$x^2 = \left(\frac{9}{y^2} - \frac{24}{y} + 16\right)$ ✓

$V = \pi \int x^2 \, dy$

$= \pi \int_6^{12} \left(9y^{-2} - \frac{24}{y} + 16\right) dy$

$= \pi \left[\frac{9y^{-1}}{-1} - 24\ln y + 16y\right]_6^{12}$ ✓

$= \left[96\frac{3}{4} - 24\ln 12 + 24\ln 6\right]\pi$ units$^3$ ✓