KILLARA HIGH SCHOOL

YEAR 12 EXTENSION 1 MATHEMATICS

TERM 2 ASSESSMENT 2010

Time allowed: 65 Minutes

Instructions

1. Show all necessary working.
2. Marks may be deducted for a lack of working or poor setting out of solutions.
3. Start each question in a new booklet.

Question 1 (15 marks) (Start a new booklet)

a) Find the value of each of the following. Give your answers in radians in exact form:

i. $\sin\left(\cos^{-1}\frac{4}{5}\right)$ 1

ii. $\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{5}{13}$ 1

b) i. State the domain and range of $y = 3\sin^{-1}2x$. 2

ii. Sketch the function $y = 3\sin^{-1}2x$. 2

c) Differentiate $2\sin^{-1}(e^{3x})$ with respect to x. 2

d) Consider the function $f(x)=\frac{4}{x^2+4}$.

Calculate the exact area bounded by the curve and the x-axis from $x=-2\sqrt{3}$ to $x=\frac{2}{3}\sqrt{3}$. 3

e) If $y=\tan^{-1}x$, then find the value of $(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}$ 2

f) Evaluate $\int\frac{dx}{x^2+4x+5}$. 2

Question 2 (16 marks) (Start a new booklet)

a) Differentiate each of the following with respect to x.

i. $e^{-2x}\sqrt{x^3}$. 2

ii. $\sin[\ln(2x-3)]$. 2

b) Integrate $\int \frac{3\sin 2x}{4+5\cos 2x}dx$ with respect to x. 2

c) Consider the function $f(x)=\frac{e^x}{3+e^x}$

i. Show that $f(x)$ has no stationary points. 2

ii. Find the coordinates of the point of inflexion.

given that $f''(x)=\frac{3e^x(3-e^x)}{(3+e^x)^3}$. 3

iii. Show that $0<f(x)<1$ for all x. 3

iv. Sketch the curve of $y=f(x)$. Show its essential features. 2

Question 3 (20 marks) (Start a new booklet)

a) A hollow cone with a vertical angle of 60^0 is held with its axis vertical and its vertex downwards. Water is poured into the cone at a rate of 12 cm^3/s. Find the rate at which the surface of the water is rising when the depth is 9cm. Give your answer as an exact value. 3

b) A particle is moving according to $x = 4\cos\frac{\pi}{4}t$, in units of metres per second.

i. Find $\dot{x}$ and $\ddot{x}$ 2

ii. Starting at $x = 0$ sketch the graph of x showing one complete period. 2

iii. What are the particles maximum displacement, velocity and acceleration and when during the first 8 seconds do they occur? 3

iv. How far does it travel during the first 20 seconds? 2

Question 3 continues over the page

Question 3 continued

c) A particle moving along the x-axis starts at the origin with an initial velocity u. Its aacceleration is given by $\frac{d^2x}{dt^2} = 4x^3 - 16x$.

i. Show that the quantity $E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - x^4 + 8x^2$ does not change with time. 3

ii. Find E if $u = \sqrt{\frac{31}{8}}$. 1

iii. Show that the particle remains at all times in the interval $|x| \le \frac{1}{2}$. 4

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

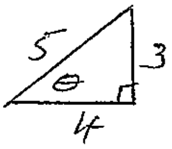
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) i

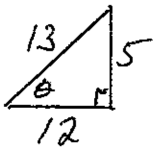


$\theta = \cos^{-1}\frac{4}{5}$

$\sin(\cos^{-1}\frac{4}{5}) = \sin\theta$

$= \frac{3}{5}$ ✓

ii



$\tan\theta = \frac{5}{12}$, $\cos(90-\theta) = \frac{12}{13}$

$\tan^{-1}(\frac{5}{12}) + \cos^{-1}(\frac{12}{13})$

$= \theta + 90 - \theta$

$= 90^\circ$ ✓

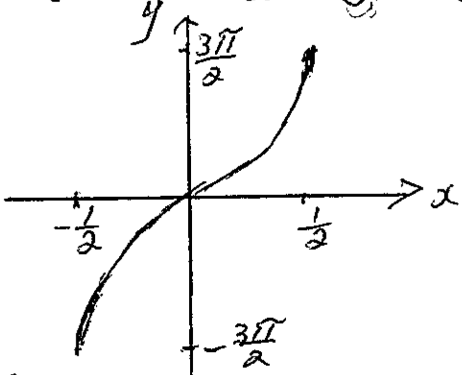
b i, Domain: $-1 \le 2x \le 1$

$y = 3\sin^{-1} x$

$-\frac{1}{2} \le x \le \frac{1}{2}$ ✓

Range: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ ✓

ii



1 for shape

1 for showing x, y-axis values

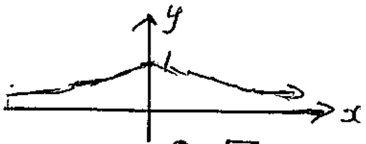
c

$\frac{d}{dx} 2\sin^{-1}(e^{3x})$

$= \frac{2}{\sqrt{1-(e^{3x})^2}} \times 3e^{3x}$ ✓

$= \frac{6e^{3x}}{\sqrt{1-e^{6x}}}$ ✓

d



Area $= \int_{-2\sqrt{3}}^{\frac{2}{3}\sqrt{3}} \left(\frac{4}{x^2+4}\right) dx$ ✓

d) Continued

Area $= 2\int_{-2\sqrt{3}}^{\frac{2}{3}\sqrt{3}} \frac{2}{2^2+x^2}\, dx$

$= \left[2\tan^{-1}\frac{x}{2}\right]_{-2\sqrt{3}}^{\frac{2}{3}\sqrt{3}}$ ✓

$= 2\tan^{-1}\frac{\sqrt{3}}{3} - 2\tan^{-1}(-\sqrt{3})$

$= 2\tan^{-1}\frac{\sqrt{3}}{3} + 2\tan^{-1}\sqrt{3}$

$= 2\pi$. ✓

e) $y = \tan^{-1} x$

$\frac{dy}{dx} = \frac{1}{1+x^2}$ $\qquad \frac{d^2y}{dx^2} = -1(1+x^2)^{-2}\cdot 2x$

$= (1+x^2)^{-1}$ $\qquad = \frac{-2x}{(1+x^2)^2}$

$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}$

$= (1+x^2)\left(\frac{-2x}{(1+x^2)^2}\right) + 2x\left(\frac{1}{1+x^2}\right)$ ✓

$= \frac{-2x}{1+x^2} + \frac{2x}{1+x^2}$

$= 0$ ✓

f $\int \frac{dx}{x^2+4x+5}$

$= \int \frac{dx}{x^2+4x+4+1}$

$= \int \frac{dx}{(x+2)^2+1}$ ✓

$= \tan^{-1}(x+2) + C$ ✓

Question 2

a i $\frac{d}{dx} e^{-2x}\sqrt{x^3}$

$= \frac{d}{dx}\left(e^{-2x}x^{\frac{3}{2}}\right)$ ✓

✓ $= \frac{3}{2}e^{-2x}x^{\frac{1}{2}} + x^{\frac{3}{2}}(-2)e^{-2x}$

$= x^{\frac{1}{2}}e^{-2x}\left[\frac{3}{2} - x^{3}\right]$

ii $\frac{d}{dx}\sin[\ln(2x-3)]$ ✓ ✓

$= \cos[\ln(2x-3)]\cdot\left(\frac{2}{2x-3}\right)$

b $\int \frac{3\sin 2x}{4+5\cos 2x}\,dx$

$= \frac{3}{2\times 5}\int \frac{-2\sin 2x}{\frac{4}{5}+\cos 2x}\,dx$ ✓

$= -\frac{3}{10}\ln(4+5\cos 2x)+c$ ✓

c i $f(x) = \frac{e^x}{3+e^x}$

$f'(x) = \frac{3e^x}{(3+e^x)^2}$ ✓

$3e^x \neq 0$ ∴ $f'(x)\neq 0$
∴ no stationary pts ✓

ii $f''(x)=0$
when $3e^x(3-e^x)=0$
ie when $e^x = 3$
ie at $(\ln 3, \frac{1}{2})$ ✓

at $(\ln 3, \frac{1}{2})$

x	1	$\ln 3$	1.2
$f''(x)$	0.01	0	−0.01
	/	—	\

✓

$f''(x)$ changes sign at $(\ln 3, \frac{1}{2})$
and $f''(x)=0$ at $(\ln 3, \frac{1}{2})$
∴ $(\ln 3, \frac{1}{2})$ is the point of inflexion. ✓

iii $f(x) = \frac{e^x}{3+e^x}$

$e^x > 0$ ✓

and $3+e^x > e^x$ ✓

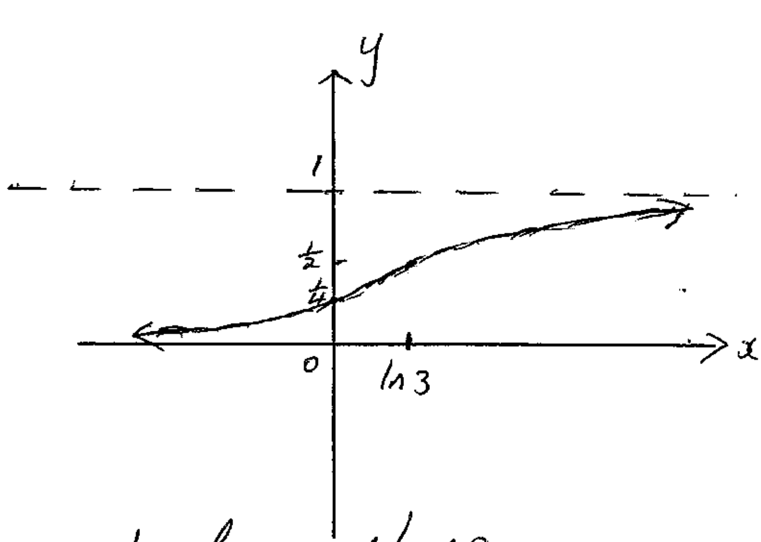
∴ $\frac{e^x}{3+e^x} < 1$

∴ $0 < f(x) < 1$ ✓

iv when $x=0$, $f(x)=\frac{1}{4}$.

$f(x) = \frac{1}{3e^{-x}+1}$

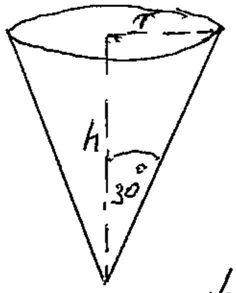
As $x\to\infty$, $e^{-x}\to 0$, $f(x)\to 1$
As $x\to-\infty$, $e^{+x}\to 0$, $f(x)\to 0$



1 for shape
1 for values (features)

Question 3

a



$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}.$

$\tan 30 = \frac{r}{h}$

$\frac{r}{h} = \frac{1}{\sqrt{3}}$

$r = \frac{h}{\sqrt{3}}$

$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$

$V = \frac{\pi}{9}h^3$ ✓

$\frac{dV}{dh} = \frac{\pi}{3}h^2$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ ✓

when $\frac{dV}{dh} = \frac{\pi}{3}h^2$, $\frac{dV}{dt} = 12$, $h = 9$

then

$12 = 27\pi \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{12}{27\pi} = \frac{4}{9\pi}$ cm/s ✓

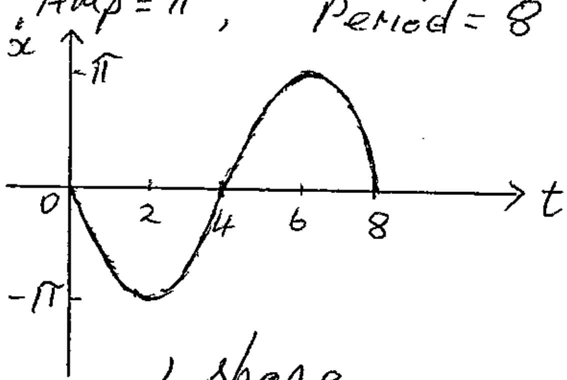
b $x = 4\cos\frac{\pi}{4}t$

i $\dot{x} = -\pi \sin\frac{\pi}{4}t$ ✓

$\ddot{x} = -\frac{\pi^2}{4}\cos\frac{\pi}{4}t$ ✓

ii $\dot{x} = -\pi \sin\frac{\pi}{4}t$

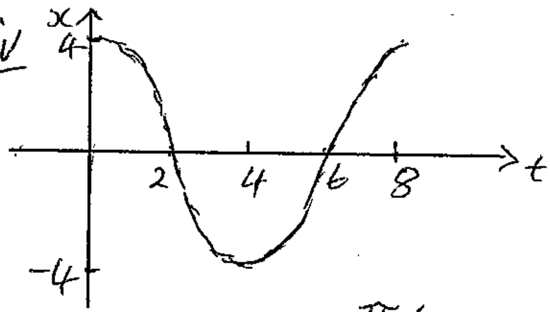
Amp $= \pi$, Period $= 8$



1 shape
1 values

iii
* Max Displace = 4 m when $t = 0$ s and $t = 8$ s ✓
* Max Vel $= \pi$ m/s when $t = 6$ s ✓
* Max accel $= \frac{\pi^2}{4}$ m/s^2 when $t = 4$ s ✓

iv



$x = 4\cos\frac{\pi}{4}t$

* from graph it travels 8m in 4 seconds ✓
* Thus travels $5 \times 8 = 40$ metres in 20 seconds ✓

c i $\frac{d^2x}{dt^2} = 4x^3 - 16x$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = 4x^3 - 16x$ ✓

$\int \frac{d}{dx}\left(\frac{v^2}{2}\right)dx = \int (4x^3 - 16x)\,dx$

$\frac{v^2}{2} = x^4 - 8x^2 + c$

$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = x^4 - 8x^2 + c$

$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - x^4 + 8x^2 = c = E$ ✓

Thus E is a constant which does not change with time. ✓

Question 3 (continued)

c

ii when $t=0$, $x=0$
and $v=u$

$E = \frac{1}{2}v^2 - x^4 + 8x^2$
$= \frac{1}{2}u^2$
$= \frac{1}{2}\left(\sqrt{\frac{31}{8}}\right)^2$
$= \frac{31}{16}$ ✓

iii $\frac{1}{2}v^2 - x^4 + 8x^2 = \frac{31}{16}$

$\frac{v^2}{2} = x^4 - 8x^2 + \frac{31}{16}$

since $v^2 \geq 0$

then $x^4 - 8x^2 + \frac{31}{16} \geq 0$ ✓

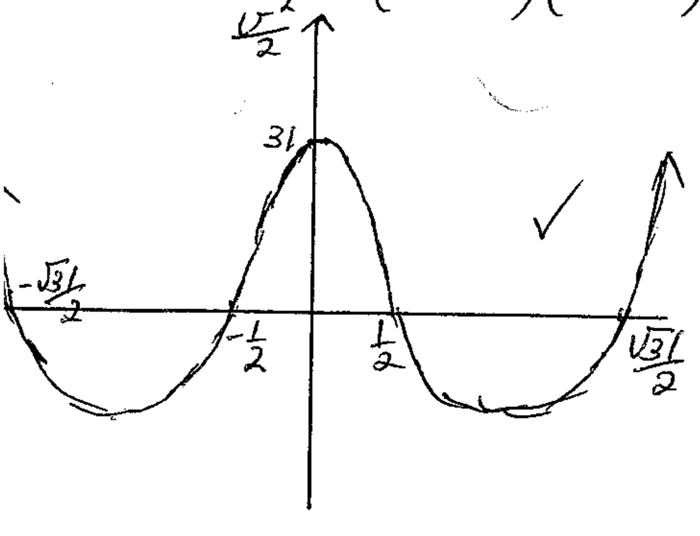
$16x^4 - 128x^2 + 31 \geq 0$

$(4x^2-1)(4x^2-31) \geq 0$

$(2x-1)(2x+1)(2x-\sqrt{31})(2x+\sqrt{31}) \geq 0$ ✓

graph of

$\frac{v^2}{2} = (2x-1)(2x+1)(2x-\sqrt{31})(2x+\sqrt{31})$



✓

However $\frac{v^2}{2} \geq 0$

this occurs when $|x| \leq \frac{1}{2}$

and $|x| \geq \frac{\sqrt{31}}{2}$

However, the particle starts at $x=0$ when $t=0$

thus the particle is in the interval $|x| \leq \frac{1}{2}$ ✓