KILLARA HIGH SCHOOL

2010

HIGHER SCHOOL CERTIFICATE
HALF YEARLY EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question

Total marks – 84

- Attempt questions 1 – 7
- All questions are of equal value

Total marks − 84

Attempt Questions 1 − 7

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 mark) Use a SEPARATE writing booklet

a) Write the exact value of $\cos \dfrac{5\pi}{4}$ **1**

b) Solve $| 2x - 3 | < 13$ **2**

c) The interval PQ has endpoints P(2, 3) and Q(−3, 5). Find the coordinates of the point which divides the interval externally in the ratio 3 : 1. **2**

d) The polynomial $P(x) = 2x^3 + ax^2 + x + 2$ has a factor $(2x + 1)$. Find the value of a. **2**

e) Find the obtuse angle between the lines $3x - y + 5 = 0$ and $2x + 3y - 1 = 0$. Give your answer to the nearest degree. **3**

f) Evaluate $\lim\limits_{x \to 0} \dfrac{2 \sin 2x}{x}$. **2**

Marks

Question 2 (12 mark) Use a SEPARATE writing booklet

a) The equation $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β, γ.

(i) Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **2**

(ii) Hence find the values of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$. **1**

b) Use the table of standard integrals to find the exact value of **2**

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx .$$

c) The variable point ($2\cos\theta$, $3\sin\theta$) lies on a curve. Find the Cartesian equation of the curve. **2**

d) Using the substitution $u = x + 2$, find $\int \frac{x}{3}\sqrt{x+2}\, dx$. **2**

e) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x + \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. **2**

(ii) Hence, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le \frac{\pi}{2}$. **1**

Marks

Question 3 (12 mark) Use a SEPARATE writing booklet

a) Use method of mathematical induction to prove that **4**

$$(1+1) + (2+3) + (3+5) + \ldots. + (n+(2n-1)) = \frac{1}{2}n(3n+1)$$

where n is a positive integer.

b) The letters of the word *MOUSE* are to be rearranged.

(i) How many arrangements are there which start with the letter *M* and and end with the letter *E*? **1**

(ii) How many arrangements are there in which the vowels are grouped together? (A vowel is one of the letters A,E,*I*,*O*,*U*) **1**

(iii) How would the arrangements in part (i) and (ii) change if the given word had been *MOOSE* instead of *MOUSE*? **2**

c) Solve: $x+2 \le \dfrac{4}{x-1}$. **4**

Marks

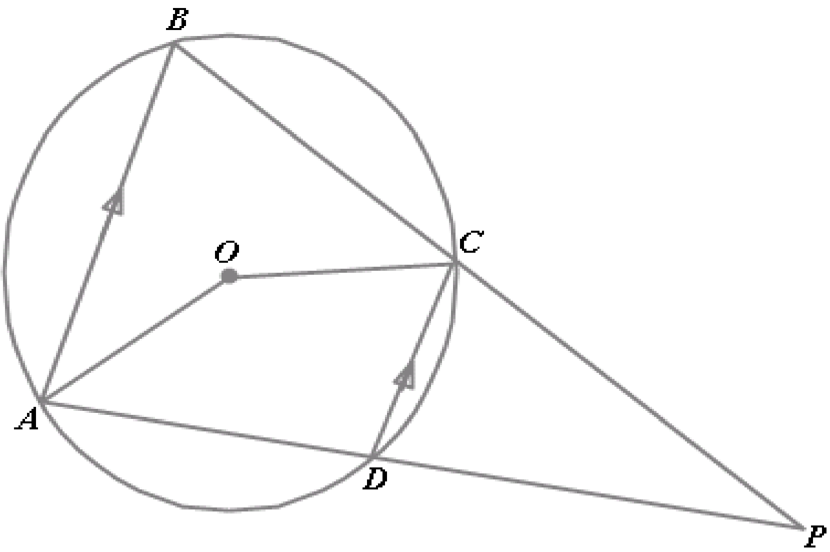
Question 4 (12 mark) Use a SEPARATE writing booklet

a) (i) Show that the equation of the normal at $P\left(2at, at^2\right)$ on the parabola $x^2 = 4ay$ is $x + ty = at^3 + 2at$. **2**

(ii) The normal intersects with the y-axis at the point Q. Find the coordinates of Q and hence the coordinates of R, the midpoint of PQ. **2**

(iii) Hence, find the Cartesian equation of the locus of R. **1**

b) In the diagram below, O is the centre of the circle and $AB \parallel CD$. AD meets BC at P.



Prove: (i) $CP = DP$. **3**

(ii) ΔABP is isosceles. **2**

(iii) $OAPC$ is a cyclic quadrilateral. **2**

Marks

Question 5 (12 mark) Use a SEPARATE writing booklet

a) Differentiate $\dfrac{\tan x}{x}$. **2**

b) Differentiate: $x \cos x$ **2**

c) Find $\int (\sin 3x + \cos 2x)\,dx$ **2**

d) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx$ **2**

e) Evaluate $\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x \, dx$ **1**

f) At any point $y = f(x)$, the gradient function is given by $\dfrac{dy}{dx} = 2\cos^2 x - 1$. **3**

If $y = \pi$ when $x = \pi$, find the value of y when $x = 2\pi$.

Marks

Question 6 (12 mark) Use a SEPARATE writing booklet

a) (i) Draw a neat sketch of $y = 3\sin 2x$ for $0 \le x \le \pi$. **2**

(ii) The area between $y = 3\sin 2x$ and, $x = 0$ and $x = \pi$ and the x - axis is rotated about the x - axis. Find the volume of the solid generated. (Leave your answer in exact form.) **3**

b) The curve $x = \dfrac{4}{\sqrt[3]{y}}$ from $y = 1$ to $y = 8$ is revolved about the y -axis. **3**

Find the volume of the solid of revolution formed.

c) (i) Show that $\dfrac{d}{dx}\left(\tan^3 x\right) = 3\sec^4 x - 3\sec^2 x$. **2**

(ii) Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$. **2**

Question 7 (12 mark) Use a SEPARATE writing booklet **Marks**

a) $y = \dfrac{(2x+1)^2}{4x(1-x)}$

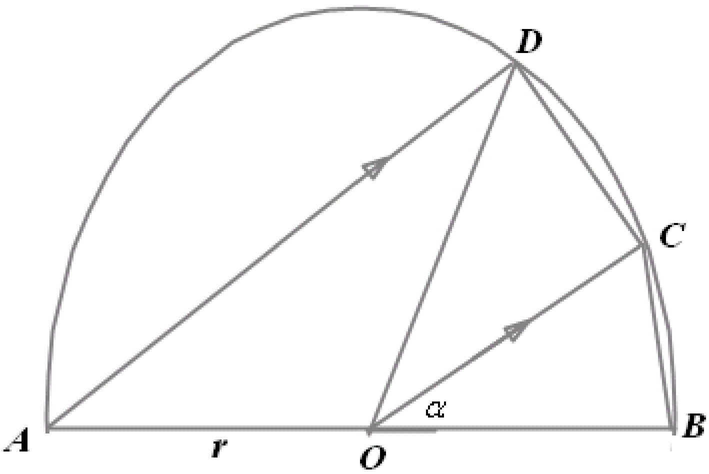
(i) Find the equations of the horizontal and vertical asymptotes of this curve **3**

(ii) Given that the curve has a relative maximum at $\left(-\dfrac{1}{2}, 0\right)$ and a relative **3**

minimum at $\left(\dfrac{1}{4}, 3\right)$, sketch the curve showing the asymptotes and turning points.

b) *ABCD* is a quadrilateral inscribed in a semi circle with centre *O* and radius *r*.

$\angle COB = \alpha$ and *AD* is parallel to *OC*.



NOT TO SCALE

(i) Show that $\angle COD = \alpha$. Give reasons. **1**

(ii) Show that the area of the quadrilateral *ABCD* can be expressed as **2**

$$A = r^2\left(\frac{1}{2}\sin 2\alpha + \sin\alpha\right)$$

(iii) Find the value of α which will produce the quadrilateral of maximum area. **3**

End of Paper

Question 1

(a)

Criteria	Marks
• Gives the correct exact value of the trig. expression	**1**

$$\cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right)$$

$$= -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$ **(1 mark)**

(b)

Criteria	Marks
• One mark for each correct solution	**2**

$|2x - 3| < 13$

$-13 < 2x - 3 < 13$
$-10 < 2x < 16$
$-5 < x < 8$

(c)

Criteria	Marks
• Substitutes correctly into the formula • Correctly evaluates the coordinates	**1** **1**

P(2, 3) and Q(–3, 5).
3 : 1

The point is $\left(\dfrac{3\times -3 - 1\times 2}{3-1}, \dfrac{3\times 5 - 1\times 3}{3-1}\right) = \left(\dfrac{-11}{2}, 6\right)$

(d)

Criteria	Marks
• Substitutes $x = -\frac{1}{2}$ into $P(x) = 0$	**1**
• Correctly evaluates the value of a	**1**

$P(x) = 2x^3 + ax^2 + x + 2$

$P\left(-\frac{1}{2}\right) = 0$ **(1 mark)**

$2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 2 = 0$

$-\frac{1}{4} + \frac{1}{4}a - \frac{1}{2} + 2 = 0 \quad \therefore\ a = -5$ **(1 mark)**

(e)

Criteria	Marks
• Evaluates the correct values of m from each expression	**2**
• Correctly uses the formula and rounds off to the nearest degree	**1**

$3x - y + 5 = 0 \Rightarrow y = 3x + 5 \quad \therefore\ m_1 = 3$ **(1 mark)**

$2x + 3y - 1 = 0. \Rightarrow y = -\frac{2}{3}x + \frac{1}{2} \quad \therefore\ m_2 = -\frac{2}{3}$ **(1 mark)**

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3 - -\frac{2}{3}}{1 + 3 \times -\frac{2}{3}}\right| = \frac{11}{3}$$

$\therefore\ \theta = 74°45'$

$\therefore$ the obtuse angle is $180 - 74°45' \approx 105°$ **(1 mark)**

(f)

Criteria	Marks
• Gives the correct solution	**2**
• Some attempt is made to manipulate the expression and identifies $\lim\limits_{x\to 0}\frac{\sin 2x}{2x} = 1$	**1**
• Bald answer	**0**
• $2 \times 2 \lim\limits_{x\to 0}\frac{\sin x}{x}$ (No fudging allowed)	**0**

$$\lim_{x\to 0}\frac{2\sin 2x}{x}. = \lim_{x\to 0} 2 \times 2\frac{\sin 2x}{2x} = 2 \times 2 \lim_{x\to 0}\frac{\sin 2x}{2x} = 2 \times 2 \times 1 = 4$$

Question 2

Outcomes assessed: E2, E4

Criteria	Marks
• Both answers correct	**2**
• One of them correct	**1**

a) (i) $x^3 - 2x^2 + 4x - 5 = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = 4$ **(1 mark)**

and $\alpha\beta\gamma = -(-5) = 5$ **(1 mark)**

Criteria	Marks
• Uses the answers from part (i) to correctly evaluate the solution	**1**

(ii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma} = \dfrac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} = \dfrac{4}{5}$

b)

Criteria	Marks
• Correctly identifies the integral of $\dfrac{1}{\sqrt{1-x^2}}$ as $\sin^{-1}x$	**1**
• Correctly evaluates the solution	**1**

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \left[\sin^{-1}(x)\right]_0^1 \quad \textbf{(1mark)}$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \textbf{(1 mark)}$$

c)

Criteria	Marks
• Gives correct expressions in terms of x and y for sinθ and cosθ	**1**
• Correctly eliminates sinθ and cosθ	**1**

$X = 2\cos\theta \quad \rightarrow \quad \cos\theta = \dfrac{x}{2}$

$Y = 3\sin\theta \quad \rightarrow \quad \sin\theta = \dfrac{y}{3}$

$\text{Cos}^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

d)

Criteria	Marks
• Uses the substitution to convert the integrand in terms of u.	**1**
• Correct integration (ignore constant of integration)	**1**

$u = x + 2 \quad \rightarrow \quad x = u - 2$

$dx = du$

$$\int \frac{x}{3}\sqrt{x+2}\,dx = \int \frac{u-2}{3}\sqrt{u}\,du \qquad \textbf{(1 mark)}$$

$$= \frac{1}{3}\int u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\,du$$

$$= \frac{1}{3}\left[\frac{2}{5}u^{\frac{5}{2}} - 2\times\frac{2}{3}u^{\frac{3}{2}}\right] + C$$

$$= \frac{2}{15}(x+2)^{\frac{5}{2}} - \frac{4}{9}(x+2)^{\frac{3}{2}} + \text{C} \qquad \textbf{(1 mark)}$$

e) (i)

Criteria	Marks
• Correctly evaluates the R	**1**
• Correctly evaluates the auxiliary angle α and gives the correct expression $2\cos\left(x + \frac{\pi}{6}\right)$	**1**

$$\sqrt{3}\cos x - \sin x \equiv R\cos(x + \alpha)$$
$$= R[\cos x \cos\alpha - \sin x \sin\alpha\ )$$

$\therefore R\cos\alpha = \sqrt{3}$

$R\sin\alpha = 1$

$\therefore R = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2, \quad R > 0.$

$\therefore \cos\alpha = \dfrac{\sqrt{3}}{2}$ and $\sin\alpha = \dfrac{1}{2}$

$\therefore \tan\alpha = \dfrac{1}{\sqrt{3}} \qquad \therefore \alpha = \dfrac{\pi}{6}, \quad 0 < \alpha < \dfrac{\pi}{2}$

$\therefore \quad \sqrt{3}\cos x - \sin x \equiv 2\cos\left(x + \dfrac{\pi}{6}\right)$

(ii)

Criteria	Marks
• Solves the trigonometric expression correctly for $x + \frac{\pi}{6}$	**1**
• Gives $\frac{\pi}{6}$ as the only solution, *must* mention the domain $0 \le x \le \frac{\pi}{2}$.	**1**

$\sqrt{3}\cos x - \sin x = 1$

$\therefore 2\cos\left(x + \dfrac{\pi}{6}\right) = 1$

$\cos\left(x + \dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$\therefore x + \dfrac{\pi}{6} = \dfrac{\pi}{3}, \quad 2\pi - \dfrac{\pi}{3} = \dfrac{\pi}{3}, \quad \dfrac{5\pi}{3}$ **(1 mark)**

$\therefore x = \dfrac{\pi}{3} - \dfrac{\pi}{6}, \quad \dfrac{5\pi}{3} - \dfrac{\pi}{6}$

$= \dfrac{\pi}{6}$ as the only solution as $0 \le x \le \dfrac{\pi}{2}$. **(1 mark)**

Question 3

a)

Criteria	Marks
• Correctly proves the result for n = 1 • Correct statement for n = k • Adds $(k+1+(2(k+1)-1))$ to the LHS of the expression for n = k to give the expression for n = k+1 • Correct algebraic techniques to produce the RHS as $\frac{1}{2}(k+1)(3(k+1)+1)$. *–1 mark without the correct conclusion*	**1** **1** **1** **1**

Let $S_n = (1+1) + (2+3) + (3+5) + \ldots + (n+(2n-1)) = \frac{1}{2}n(3n+1)$

Step 1: Prove S_1 is true.

$S_1 = (1+1) = \frac{1}{2}\times 1(3\times 1+1)$

$2 = 2$

LHS = RHS

$\therefore$ S_1 true. **(1 mark)**

Step2 : Assume S_k is true.

$S_k = (1+1) + (2+3) + (3+5) + \ldots + (k+(2k-1)) = \frac{1}{2}k(3k+1)$

Step2 : Prove S_{k+1} is true.

LHS $(1+1) + (2+3) + (3+5) + \ldots + (k+(2k-1)) + (k+1+(2(k+1)-1))$ **(1 mark)**

$$= \frac{1}{2}k(3k+1) + (k+1+(2(k+1)-1))$$

$$= \frac{1}{2}k(3k+1) + (k+1)+(2k+1)$$

$$= \frac{1}{2}[k(3k+1)]+2(3k+2)$$

$$= \frac{1}{2}[3k^2+7k+4]$$

$$= \frac{1}{2}(k+1)(3k+4)$$

$$= \frac{1}{2}(k+1)(3(k+1)+1)$$ **(1 mark)**

$\therefore$ S_{k+1} is true, given that S_k is true.

Conclusion: We have proved S_1 is true and then using the result from step 3, S_2 is true; hence S_3 is true and so on S_n is true for all integral values of *n*.

b) (i)

Criteria	Marks
• Correctly evaluates the number of arrangements	**1**

No. of arrangements =

$\boxed{\mathbf{M}}\ \underline{3}\ \ \underline{2}\ \ \underline{1}\ \boxed{\mathbf{E}} = 6$

(ii)

Criteria	Marks
• Correctly evaluates the number of arrangements	**1**

Consider OUE as one group. $\boxed{\mathbf{M}}\ \boxed{\mathbf{OUE}}\ \boxed{\mathbf{S}}$

The three letters can be organised in 3! Ways and $\boxed{\mathbf{OUE}}$ can be organised in 3! Ways.
$\therefore$ total number of arrangements = $3! \times 3! = 36$ ways

(iii)

Criteria	Marks
• Correctly evaluates the number of arrangements and concludes that the no. of arrangements in part (i) is halved	**1**
• Correctly evaluates the number of arrangements and concludes that the no. of arrangements in part (ii) is halved	**1**

$\boxed{\mathbf{M}}\ \underline{\mathbf{O}}\ \ \underline{\mathbf{O}}\ \ \underline{\mathbf{S}}\ \boxed{\mathbf{E}}$

No. of arrangements = $\dfrac{3!}{2!} = 3$ $\therefore$ Answer in part (i) is halved.

$\underline{\mathbf{M}}\ \boxed{\mathbf{OOE}}\ \underline{\mathbf{S}}$

No. of arrangements = $3! \times \dfrac{3!}{2!} = 18$. The answer in part (ii) is halved.

c)

Criteria	Marks
• Multiplies both sides of the inequality by $(x-1)^2$ • Correctly manipulates the expression to get $(x-1)(x^2+x-6) \le 0$ • Solves the inequality to get $x \le -3$; $1 \le x \le 2$, $x \ne 1$ • Excludes $x = 1$, to get $x \le -3$; $1 < x \le 2$	1 1 1 1

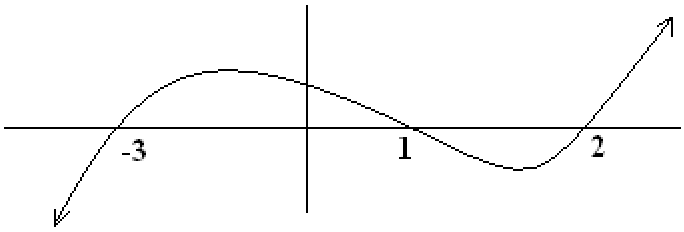
$x + 2 \le \dfrac{4}{x-1}$.

Multiply by $(x-1)^2$,

$(x+2)(x-1)^2 \le 4(x-1)$ **(1 mark)**

$(x+2)(x-1)^2 - 4(x-1) \le 0$

$(x-1)\left[(x+2)(x-1)-4\right] \le 0$



$(x-1)(x^2+x-2-4) \le 0$

$(x-1)(x^2+x-6) \le 0$

$(x-1)(x+3)(x-2) \le 0$

$\therefore\ x \le -3;\quad 1 \le x \le 2,\quad x \ne 1$

Solution is $x \le -3;\quad 1 < x \le 2$

Question 4

a)

Criteria	Marks
• Correctly derives the gradient of the normal (*must show all steps*) • Correctly derives the equation of the normal using the gradient from above.	1 1

$x = 2at \qquad y = at^2$

$\therefore \dfrac{dy}{dx} = \dfrac{2at}{2a} = t.$

$\therefore$ Gradient of the normal is $-\dfrac{1}{t}$. **(1 mark)**

Equation of the normal is

$y \quad - \quad = -\dfrac{1}{t}\,(x - 2at)$

$t(y - at^2) = -x + 2at$

$\therefore x + ty = at^3 + 2at$. **(1 mark)**

(ii)

Criteria	Marks
• Substitutes y = 0 into the equation of the normal to find Q. • Correctly proves the coordinates of *R*.	1 1

x-intercept: put $x = 0$

$0 + ty = at^3 + 2at.$

$\therefore y = 2a + at^2$

$\therefore$ Q $(0, 2a + at^2)$ **(1 mark)**

$P\left(2at, at^2\right)$

$\therefore$ The midpoint $R \left(\dfrac{0 + 2at}{2}, \dfrac{2a + at^2 + at^2}{2}\right)$

$R(at, \; a + at^2) = R\,[at, a(1 + t^2)]$

(iii)

Criteria	Marks
• Sets $x = a(1 + t^2)$ and $y = at$ and eliminates t to the Cartesian equation of the locus of R.	**1**

$y = a(1 + t^2)$

$x = at \Rightarrow t = \dfrac{x}{a}$

$\therefore\ y = a\left(1 + \left[\dfrac{x}{a}\right]^2\right)$

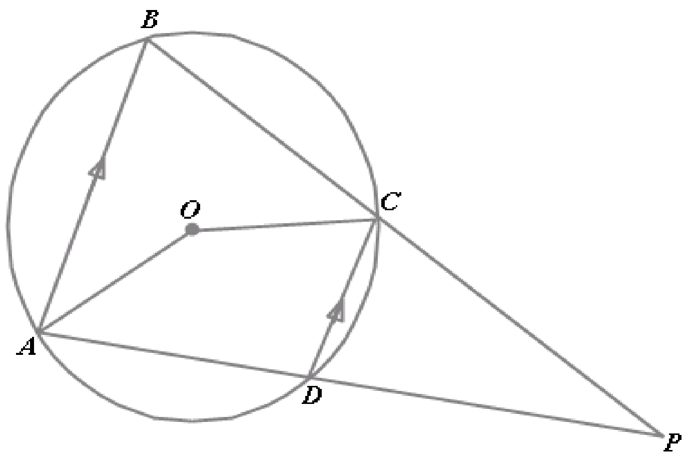
$= a\left(1 + \dfrac{x^2}{a^2}\right)$

$y = a + \dfrac{x^2}{a}$

$y - a = \dfrac{y^2}{a}$ $\quad\therefore\ x^2 = a(y - a)$ is a parabola with focal length $\dfrac{a}{4}$ and vertex $(0,a)$

b)

Criteria	Marks
• One mark for each step of working with correct reasoning	**3**



(i) Let $\angle PDC = \alpha$

$\therefore\ \angle PDC = \angle ABC = \alpha$ 1

(the exterior angle of a cyclic quadrilateral equals the sum of the opposite interior angles **(1 mark)**

$\angle ABC = \angle DCP = \alpha$ ($AB \parallel DC$, corresponding angles in parallel lines are equal) 2

(1 mark)

ie. $\angle PDC = \angle DCP = \alpha$ (using 1 and 2) 3

$\therefore$ PD = PC (using 3 ΔPCD is isosceles and therefore the sides are equal) **(1 mark)**

(ii)

Criteria	Marks
• One mark for each step of working with correct reasoning	**2**

$\angle PDC = \alpha$
$\therefore \angle PDC = \angle PAB = \alpha$ (CD || AB, corresponding angles in parallel lines are equal) **(1 mark)**
$\therefore \angle PBA = \angle PAB = \alpha$ (base angles are equal, $\therefore$ ΔPAB is isosceles) **(1 mark)**

(iii)

Criteria	Marks
• One mark for each step of working with correct reasoning	**2**

$\angle ABC = \alpha$
$\therefore \angle AOC = 2\alpha$ (angle subtended by arc AC at the centre
is double of what is subtended at the circumference.) **(1 mark)**

In ΔPCD,

$$
\begin{aligned}
\angle CPD &= 180 - (\angle PCD + \angle PDC) \\
&= 180 - (\alpha + \alpha) \\
&= 180 - 2\alpha \quad \text{(angle sum of triangle add to } 180^\circ \text{)}
\end{aligned}
$$

$\therefore$ In quadrilateral OAPC,
$\angle CPA + \angle AOC = 180 - 2\alpha + 2\alpha = 180^\circ$

$\therefore$ OAPC is cyclic. (opposite angles add to 180°) **(1 mark)**

Question 5

a)

Criteria	Marks
• Correct differentiation of $\tan x$ and x • Applies quotient rule correctly	1 1

$$y = \frac{\tan x}{x}$$

$$u = \tan x \qquad u' = \sec^2 x$$
$$v = x \qquad v' = 1$$
$$\frac{dy}{dx} = \frac{x\sec^2 x - \tan x}{x^2}$$

b)

Criteria	Marks
• Correct differentiation of $\cos x$ and x • Applies product rule correctly	1 1

$$y = x\cos x$$

$$u = x \qquad u' = 1$$
$$v = \cos x \qquad v' = -\sin x$$
$$\frac{dy}{dx} = -x\sin x + \cos x$$

c)

Criteria	Marks
• Correct integration of $\sin 3x$ and $\cos 2x$	2

$$\int (\sin 3x + \cos 2x)\,dx = \frac{-\cos 3x}{3} + \frac{\sin 2x}{2} + C$$

d)

Criteria	Marks
• Correct integration • Evaluates the expression correctly	1 1

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos x\,dx = \left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{3}[\,1-0\,] = \frac{1}{3}$$

e)

Criteria	Marks
• Uses the property of odd functions to evaluate correctly	**1**

$$\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x \, dx$$

$\cos^2 x \sin^3 x$ is an odd function; $\therefore \int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x \, dx = 0$

f)

Criteria	Marks
• Integrates $\frac{dy}{dx} = \cos 2x$ correctly	**1**
• Evaluates the constant of integration by substitution	**1**
• Obtains the primitive function and substitutes $x = 2\pi$ to give correct value of y.	**1**

$$\frac{dy}{dx} = 2\cos^2 x - 1$$

ie. $\frac{dy}{dx} = \cos 2x$.

$\therefore \; y = \frac{\sin 2x}{2} + C$ **(1 mark)**

When $x = \pi$ when $y = \pi$.

$\therefore \; \pi = 0 + C \Rightarrow C = \pi$ **(1 mark)**

$\therefore y = \frac{\sin 2x}{2} + \pi$.

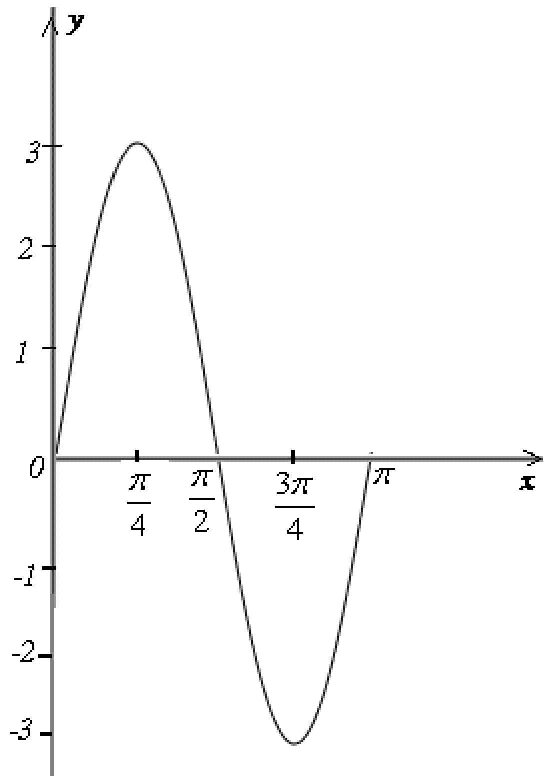
When $x = 2\pi$, $y = 0 + \pi$ $\therefore y = \pi$ **(1 mark)**

Question 6

a)(i)

Criteria	Marks
• Correct amplitude and period • Correct shape and uniform scale on the axes and correct labels	1 1

Amplitude = 3 and period $\frac{2\pi}{2} = \pi$



(ii)

Criteria	Marks
• Correct expression to calculate the volume • Rewrites $\sin^2 2x$ as $\frac{1-\cos 4x}{2}$ • Correct integration and evaluation	1 1 1

Volume $= 2 \times \pi \int_0^{\frac{\pi}{2}} (3\sin 2x)^2 dx$

$= 2 \times \pi \times 9 \times \int_0^{\frac{\pi}{2}} \sin^2 2x dx$ **(1 mark)**

$= 2 \times \pi \times 9 \times \int_0^{\frac{\pi}{2}} \frac{1-\cos 4x}{2} dx$ **(1 mark)**

$= 9\pi \int_0^{\frac{\pi}{2}} 1 - \cos 4x \, dx$

$= 9\pi \left[x - \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{2}}$

$= 9\pi \left[\frac{\pi}{2} - \frac{1}{4} \times 0 - 0 \right]$

$= \frac{9\pi^2}{2}$ unit^2. **(1 mark)**

b)

Criteria	Marks
• Makes x^2 subject of the formula correctly. • Gives the expression for volume with y-axis $\pi \int_1^8 16 y^{-\frac{2}{3}} dy$ • Correct integration and evaluation	**1** **1** **1**

$x \sqrt[3]{y} = 4$ from $y = 1$ to $y = 8$

$\therefore$ x is positive. The graph is in the 1st quadrant.

Volume $= \int \pi x^2 dy$

$x \sqrt[3]{y} = 4$

$x = \frac{4}{\sqrt[3]{y}}$

$\therefore x^2 = \frac{16}{y^{\frac{2}{3}}}$ **(1 mark)**

$\therefore$ Volume $= \pi \int_1^8 16 y^{-\frac{2}{3}} dy$ **(1 mark)**

$= 16\pi \times 3 \times \left[y^{\frac{1}{3}} \right]_1^8$

$= 48\pi [2 - 1]$

$= 48\pi$ unit^3. **(1 mark)**

c)(i)

Criteria	Marks
• Differentiates $\tan^3 x$ correctly. • Substitutes $\tan^2 x$ as $\sec^2 x - 1$ and completes the proof	**1** **1**

$\frac{d}{dx}\left(\tan^3 x\right) = 3 \tan^2 x . \sec^2 x$ **(1 mark)**

$= 3 (\sec^2 x - 1) \sec^2 x$

$= 3 \sec^4 x - 3 \sec^2 x$ **(1 mark)**

(ii)

Criteria	Marks
• Correct solution	**2**
• Demonstrates the understanding that integration is the opposite operation of differentiation	**1**

Using the result from part (i),

$$\therefore \int 3\sec^4 x - 3\sec^2 x dx = \tan^3 x$$

$$\therefore \int \sec^4 x dx - \int \sec^2 x dx = \frac{\tan^3 x}{3}$$ **(1 mark)**

$$\int \sec^4 x dx - \tan x = \frac{\tan^3 x}{3}$$

$$\int_0^{\frac{\pi}{4}} \sec^4 x\, dx = \left[\frac{\tan^3 x}{3} + \tan x\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{3} + 1\right) - 0$$

$$\frac{4}{3}$$ **(1 mark)**

Question 7

a) (i)

Criteria	Marks
• One mark for each of the asymptotes	**3**

$$y = \frac{(2x+1)^2}{4x(1-x)}$$

Vertical asymptotes : $x = 0$ and $x = 1$ **(1 mark each)**

Horizontal asymptotes : $y = \dfrac{(2x+1)^2}{4x(1-x)}$

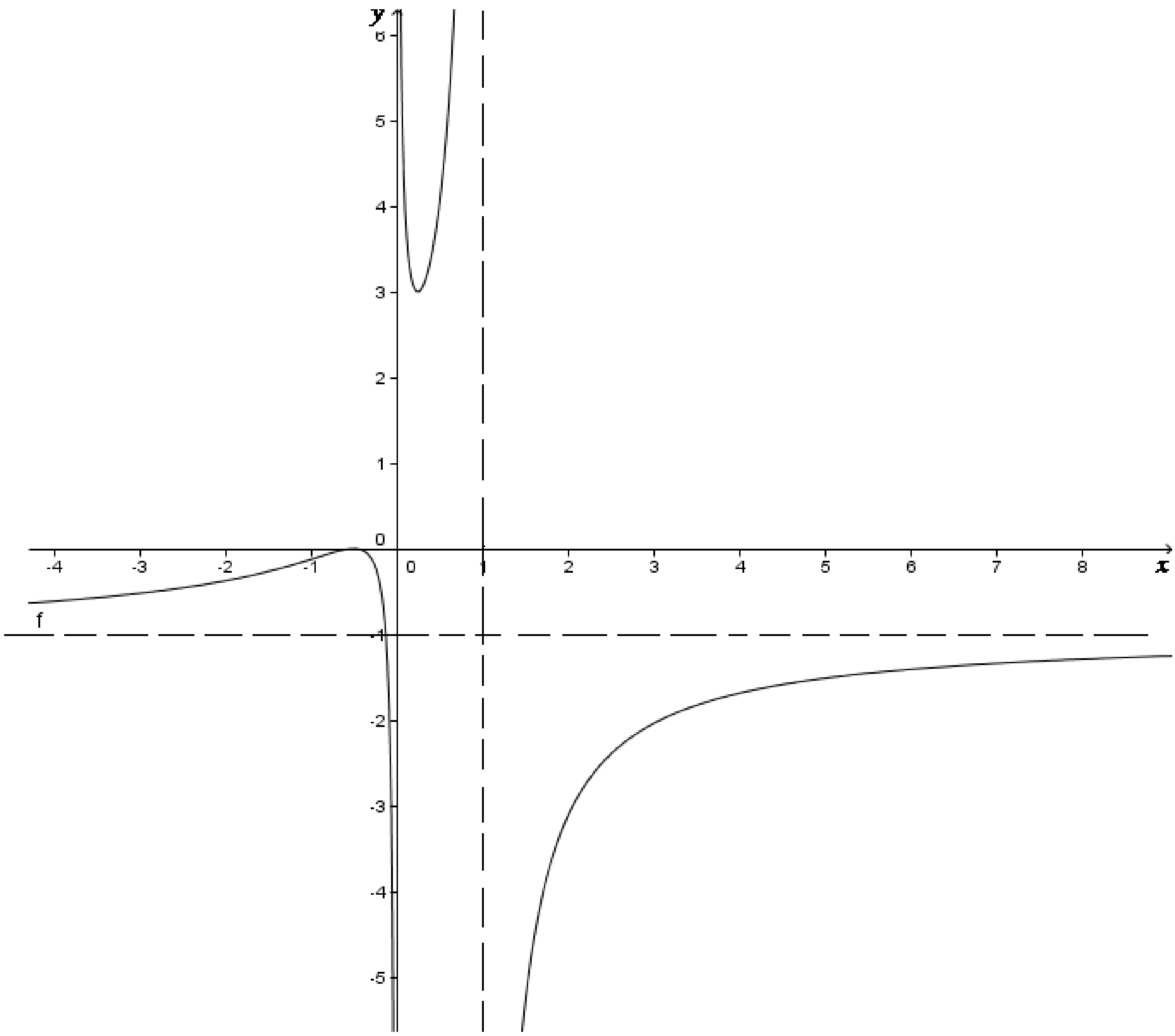
$$y = \frac{(4x^2 + 4x + 1)}{4x - 4x^2}$$

$$\lim_{x \to \pm\infty} \frac{4 + \frac{4}{x} + \frac{1}{x^2}}{\frac{4}{x} - 4} \to -1$$

$\therefore$ $y = -1$ is an asymptote. **(1 mark)**

(ii)

Criteria	Marks
• Appropriate scale and shape • Asymptotes drawn correctly • Turning points labelled correctly	**1** **1** **1**



b) (i)

Criteria	Marks
• Produces the table of values correctly • Applies Simpson's rule and rounds off correctly	**1** **1**

$\angle COB = \alpha$. $\therefore$ $\angle OAD = \alpha$. (Corresponding angles in parallel lines are equal $AB \parallel CD$)

$AO = OD = r$ (equal radii)

$\Delta\, OAD$ isosceles (2 equal sides)

$\therefore \angle\, ADO = \alpha$ (base angles of isosceles triangle are equal)

$\therefore \angle COD = \alpha$. Alternate angles in parallel lines are equal $AB \parallel CD$)

(ii) Area of quadrilateral ABCD = Area of ΔAOD + Area of ΔCOD + Area of ΔBOC

$\angle AOD = 180 - 2\alpha$.

Area of ΔAOD = $\frac{1}{2}r^2 \sin(180 - 2\alpha) = \frac{1}{2}r^2 \sin 2\alpha$

Area of ΔCOD = Area of ΔBOC $= \frac{1}{2}r^2 \sin\alpha$

$\therefore$ Area of quadrilateral ABCD, $A = \frac{1}{2}r^2 \sin 2\alpha + 2 \times \frac{1}{2}r^2 \sin\alpha$

$$A = r^2\left(\frac{1}{2}\sin 2\alpha + \sin\alpha\right)$$

(iii) Differentiating A with respect to α to find the maximum area,

$$\frac{dA}{d\alpha} = r^2\left(\frac{1}{2}\times 2\cos 2\alpha + \cos\alpha\right) = r^2(\cos 2\alpha + \cos\alpha)$$

But $\cos 2\alpha = 2\cos^2\alpha - 1$

$$\therefore \frac{dA}{d\alpha} = r^2(2\cos^2\alpha - 1 + \cos\alpha)$$

$$= r^2(2\cos^2\alpha + \cos\alpha - 1)$$

Let $\frac{dA}{d\alpha} = 0$ for possible stationary points.

$$\therefore r^2(2\cos^2\alpha + \cos\alpha - 1) = 0$$

$$(2\cos^2\alpha + \cos\alpha - 1) = 0$$

$$(2\cos\alpha - 1)(\cos\alpha + 1) = 0$$

$$\therefore \cos\alpha = \frac{1}{2}, -1$$

$\therefore \alpha = \frac{\pi}{3}, \pi$, but $0 < \alpha < \pi$

$$\therefore \alpha = \frac{\pi}{3}$$

α	1	$\frac{\pi}{3}$	2
$\frac{dA}{d\alpha}$	$0.12r^2$	0	$-0.16r^2$
	/	—	\

The maximum area of the quadrilateral occurs when $\alpha = \frac{\pi}{3}$ radians.