Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS

Extension 1

ASSESSMENT TASK 1

Date: 06/12/2011

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 60 minutes

INSTRUCTIONS: **SECTION A:** 4 Multiple Choice Questions: Circle your correct choice

SECTION B: ANSWER all 3 questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

16.1 Graphs of simple polynomials.
16.2 The remainder and factor theorems.
16.3 The roots and coefficients of a polynomial equation.
16.4 Iterative methods for numerical estimation of the roots of a polynomial equation
7.4 Mathematical induction. Applications
2.8 Simple angle properties of a circle.
2.9 Derivation of further angle, chord and tangent results.
2.10 Applications of 2.2, 2.3, 2.7, 2.8 and 2.9 to numerical and theoretical problems requiring one or more steps of reasoning.

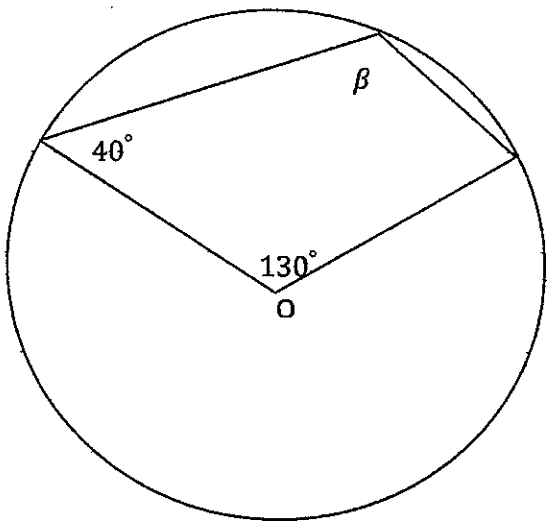
	MARKS
SECTION A Multiple Choice questions	/5
SECTION B	
QUESTION 1 Polynomials	/15
QUESTION 2 Circle Geometry	/10
QUESTION 3 Mathematical Induction	/5
	/35

This assessment task constitutes 10% of the Higher School Certificate Course Assessment.

SECTION I

Marks

1) 1



O is the centre of the circle. The value of β is:

A) 115°

B) 140°

C) 65°

D) 50°

2) If the roots of the equation $x^3 - 8 = 0$ are α, β and γ, then the value of $\alpha^3 + \beta^3 + \gamma^3$ is 1

A) 8

B) 6

C) 24

D) $\frac{8}{3}$

Marks

3) Which of the following are true about a cubic function: 1

I It must have at least one real zero

II If it has only one real zero then it must be monotonic

III If it has a triple root then it must be monotonic

A) I and II only

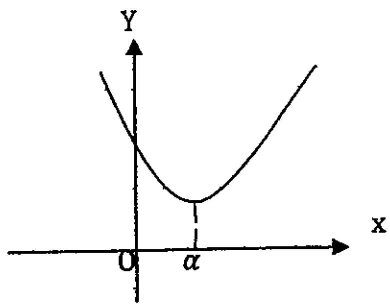
B) I and III only

C) II and III only

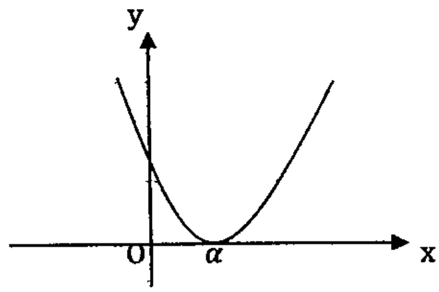
D) All I, II and III

4) A polynomial $f(x)$ has a multiple zero at $x = \alpha$. 1

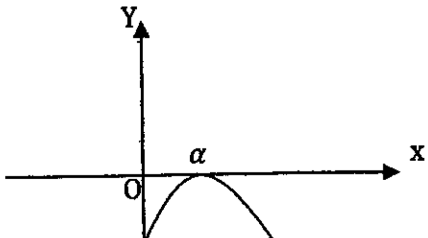
Which of the following is a possible graph of $y = f(x)$?



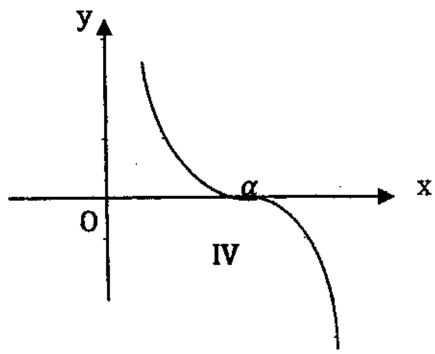
I



II



III



IV

A) I or II

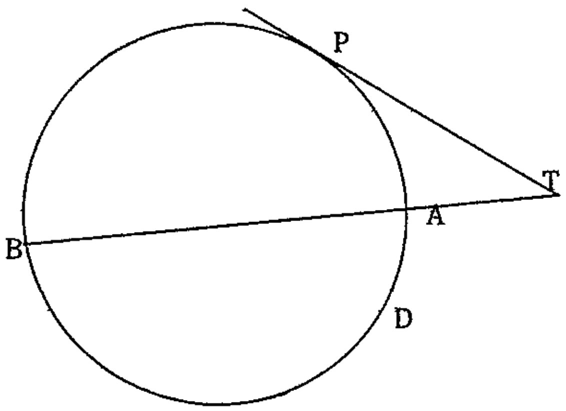
B) II or III

C) III or IV

D) II or III or IV

Marks

5) 1



TP is a tangent to the circle at P. TAB is a straight line passing through the centre of the circle. If PT = 8 cm and TA = 4 cm, then the radius of the circle is:

A) 10 cm

B) 9 cm

C) 6 cm

D) 12 cm

End Of Section I

SECTION B

Question 1 (Use a separate writing booklet) **Marks**

a) A polynomial $P(x)$ is given by $P(x) \equiv x^3 + 3x^2 - 2x + 1$.

i). Find the remainder when $P(x)$ is divided by $(x+2)(x-3)$. 2

ii). Hence, find the values of p and q, if the polynomial

$f(x) \equiv x^3 + 3x^2 + px + q$ is exactly divisible by $(x+2)(x-3)$. 2

b) The roots of the equation $3x^3 - 16x + 9 = 0$ are α, β and γ.

Find the values of:

i). $\alpha\beta\gamma$ 1

ii). $\alpha\beta + \beta\gamma + \gamma\alpha$ 1

iii). $\frac{2}{\alpha+\beta} + \frac{2}{\beta+\gamma} + \frac{2}{\gamma+\alpha}$ 2

c)

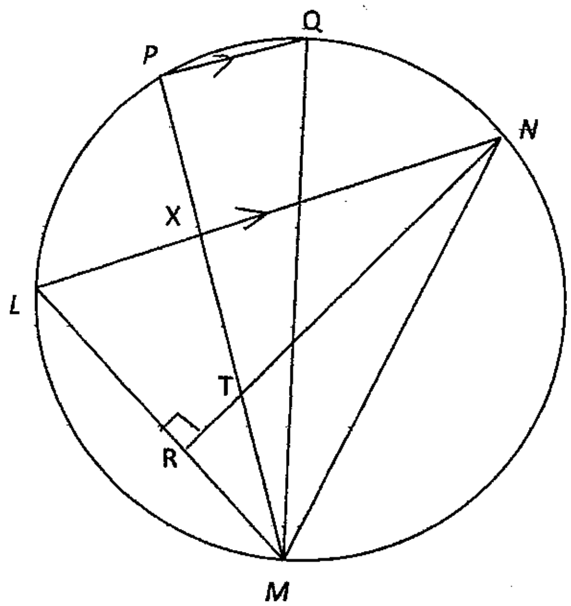
i). α is a zero of $f(x) = 3x^3 - 16x + 9$. Show that $1 < \alpha < 2$. 2

ii). Find the turning point of $y = f(x)$ between 1 and 2. 2

iii). David decided to apply Newton's method to find an approximation for α. 3
He chose 1.2 as his first approximation.
Explain to David with the aid of diagrams why the method would fail.
What range of values should David choose as his first approximation so that he will get a better second approximation?.

Question 2 (Use a separate writing booklet) **Marks**

a) MQ is a diameter of the circle drawn below. $PQ//LN$. *PM* cuts *LN* at *X*. $NR \perp LM$ and cuts *PM* at *T*.



Copy or trace the diagram into your booklet.

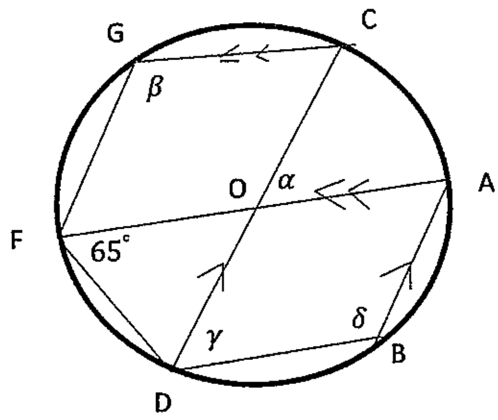
i). Explain why *LXTR* is a cyclic quadrilateral. **2**

ii). Prove, giving reasons, that: $\angle NMQ = \angle PML$ **3**

iii). Deduce that $PL = QN$ **1**

Marks

b) 4



O is the centre of the circle. $DC//BA$ and $AF//CG$. $\angle OFD=65°$ find the values, giving reasons, of the angles denoted by the pronumerals α, β, γ and δ

Question 3 (Use a separate writing booklet)

i). Prove by mathematical induction that 4

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+3)} = \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

for all integers $n \geq 0$.

ii). Deduce the value of $\sum_{r=0}^{\infty} \frac{1}{(r+1)(r+3)}$ 1

End of Paper

Section II

Question 1 a (i) $(x+2)(x-3) = x^2 - x - 6$

(a) Method 1

$$\begin{array}{r|l} & x + 4 \\ \hline x^2 - x - 6 & x^3 + 3x^2 - 2x + 1 \\ & x^3 - x^2 - 6x \\ \hline & \quad 4x^2 + 4x + 1 \\ & \quad 4x^2 - 4x - 24 \\ \hline & \qquad 8x + 25 \end{array}$$

$\therefore$ The remainder $= 8x+25$

Method 2

Divisor $(x+2)(x-3)$ is of degree 2

$\therefore$ The remainder is linear (degree 1) of the form $ax+b$ ~~ax+b~~

Let the quotient be $Q(x)$.

Writing the division transformation

$P(x) = x^3 + 3x^2 - 2x + 1 \equiv (x+2)(x-3).Q(x) + ax+b$

$\therefore P(-2) = (-2)^3 + 3(-2)^2 - 2(-2) + 1 = 0 + -2a+b$

$\Rightarrow -2a+b = 9$ —— ①

$P(3) = (3)^3 + 3(3)^2 - 2(3) + 1 = 0 + 3a+b$

$\Rightarrow 3a+b = 49$ —— ②

solving equations ① and ② simultaneously,

$a = 8, \; b = 25$

$\therefore$ The remainder $= \underline{8x+25}$

Q1 (a) (ii)

$x^3+3x^2-2x+1=(x+2)(x-3).Q(x)+8x+25$

$\Rightarrow x^3+3x^2-10x-24=(x+2)(x-3).Q(x)$

(ie) $x^3+3x^2-10x-24$ is divisible by $(x+2)(x-3)$

Since x^3+3x^2+px+q is divisible by $(x+2)(x-3)$

$x^3+3x^2+px+q=x^3+3x^2-10x-24$

matching the coefficients of corresponding terms,

$\underline{\underline{p=-10}}$ and $\underline{\underline{q=-24}}$

(b) $3x^3-16x+9=0$ roots α,β and γ

(i) $\alpha\beta\gamma=-9/3=-3$ //

(ii) $\alpha\beta+\beta\gamma+\gamma\alpha=-16/3$ //

(iii) $\dfrac{2}{\alpha+\beta}+\dfrac{2}{\beta+\gamma}+\dfrac{2}{\gamma+\alpha}$

Since $\alpha+\beta+\gamma=0$ $\alpha+\beta=-\gamma$, $\beta+\gamma=-\alpha$ and $\gamma+\alpha=-\beta$

$=-\dfrac{2}{\gamma}-\dfrac{2}{\alpha}-\dfrac{2}{\beta}$

$=-\dfrac{2}{\alpha\beta\gamma}(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=\dfrac{-2(-16/3)}{-3}$

$=\underline{\underline{-32/9}}$

Q1 (c) (i) $f(x) = 3x^3 - 16x + 9$

$f(1) = 3 - 16 + 9 = -4$

$f(2) = 24 - 32 + 9 = 1$

$f(x)$ is a cubic function.

$f(x)$ is continuous in the interval $1 \le x \le 2$

Since $f(1) < 0$ and $f(2) > 0$, there exists α such that $1 < \alpha < 2$ and $f(\alpha) = 0$

(ie) $1 < \alpha < 2$, and α is a root.

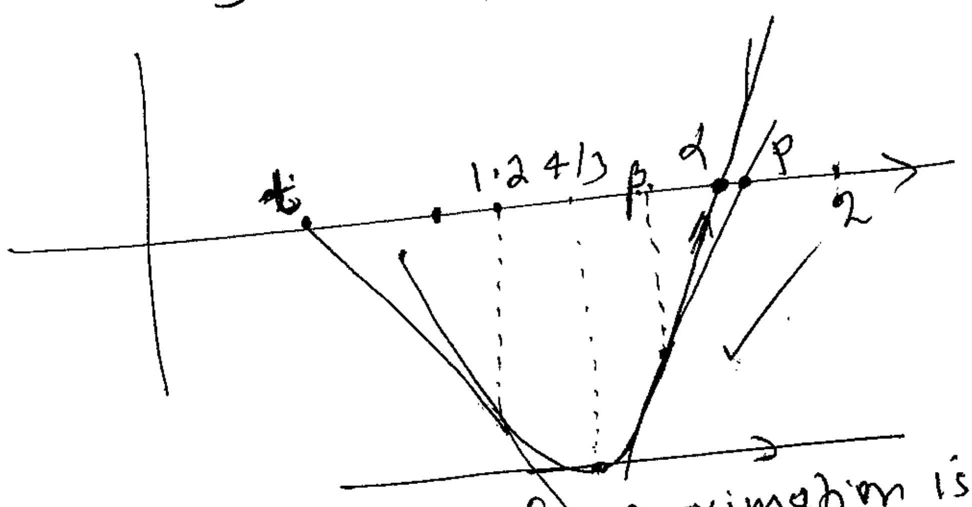
(ii) $f'(x) = 9x^2 - 16$

$f'(x) = 0 \implies x = \frac{4}{3}$ or $x = -\frac{4}{3}$

$f''(x) = 18x$

at $x = 4/3$ $f''(x) = 18 \times \frac{4}{3} > 0$ $\therefore \left(\frac{4}{3}, -\frac{47}{9}\right)$ is a minimum turning point

(iii) Below is a rough sketch of ~~y = f(x)~~ the graph of $y = f(x)$ close to the interval $[1, 2]$.



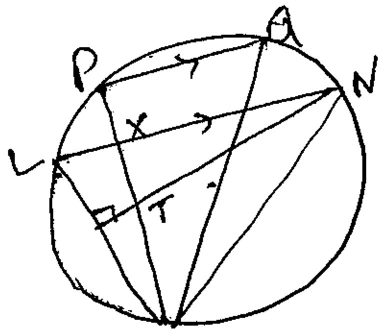
Newton's method of approximation is based on finding the x-intercept of the tangent drawn to the curve at the first approximation.

$\therefore$ If the first approximation $= 1.2$ then the second approximation is $x = t$ which is further away from ~~x~~ than 1.2

If the first approximation $= 4/3$ then the tangent at $x = 4/3$ will not have an x-intercept $\therefore$ second approximation cannot be found

If the first approximation is between $4/3$ and 2 then the second approximation is p which is a better approximation than the first at $x = \beta$.

Section II
Question 2(a)



(i) $\angle QPM = 90°$ (angle subtended by diameter MQ on the circle)

$\angle QPM = \angle PXL$ (alternate angles on $\parallel$ lines, $PQ \parallel LM$)

$\therefore \angle PXL = 90°$

$\angle TRL = 90°$ (given)

(i.e) $\angle PXL = \angle TRL$

$\therefore$ LXTR is a cyclic quadrilateral as the exterior angle PXL = interior opposite angle TRL

(ii) $\angle NMQ = \angle QLP$ (angles subtended on the circle by the same arc $\widehat{NQ}$)

But $\angle QLP = \angle PQL$ (alternate angles on $\parallel$ lines; $PQ \parallel LM$)

and $\angle PQL = \angle PML$ (angles subtended on the circle by the same arc $\widehat{LP}$)

$\Rightarrow \angle NMQ = \angle PML$

(iii) $\angle NMQ$ is the angle subtended on the circle by the chord QN

$\angle PML$ is the angle subtended on the circle by the chord PL

Since $\angle NMQ = \angle PML$ (Proven in (ii))

$PL = QN$ (angles subtended on the circle by equal chords are equal)

Q2 (b)

In ΔOFD

$OF = OD$ (radii of same circle)

$\therefore \angle OFD = \angle ODF = 65^\circ$

and $\angle FOD = 180^\circ - 2 \times 65^\circ$

$= 50^\circ$

$\angle FOD = \alpha$ vertically opposite angles

$\therefore \boxed{\alpha = 50^\circ}$

Reflex $\angle FOC = 180^\circ + \alpha$

$= 230^\circ$

$\beta = \frac{1}{2} \times$ reflex $\angle FOC$ (angle subtended by arc FDBAC on the circle is half the angle subtended at the centre by same arc)

$= \frac{1}{2} \times 230^\circ$

$\boxed{\beta = 115^\circ}$

FDBA is a cyclic quadrilateral

$\therefore \delta + 65^\circ = 180^\circ$ (opp. $\angle$s of a cyclic quad are supplementary)

$\boxed{\delta = 115^\circ}$

$\gamma + \delta = 180^\circ$ (cointerior angles on $\parallel$ lines DO $\parallel$ BA given)

$\therefore \gamma = 180^\circ - 115^\circ$

$\boxed{\gamma = 65^\circ}$

Question 3

(1) $$\sum_{r=0}^{n} \frac{1}{(r+1)(r+3)} = \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

For $n=0$ LHS $= \frac{1}{1\cdot 3} = \frac{1}{3}$

RHS $= \frac{3}{4} - \frac{1}{4} - \frac{1}{6}$

$= \frac{1}{3}$

$\therefore$ LHS $=$ RHS

$\therefore$ True for $n=0$

Assume true for $n=k$

ie $$\sum_{r=0}^{k} \frac{1}{(r+1)(r+3)} = \frac{3}{4} - \frac{1}{2(k+2)} - \frac{1}{2(k+3)}$$

for $n=k+1$

$$\sum_{r=0}^{k+1} \frac{1}{(r+1)(r+3)} = \sum_{r=0}^{k} \frac{1}{(r+1)(r+2)} + \frac{1}{(k+2)(k+4)}$$

$$= \frac{3}{4} - \frac{1}{2(k+2)} - \frac{1}{2(k+3)} + \frac{1}{(k+2)(k+4)}$$

$$= \frac{3}{4} - \frac{1}{2(k+3)} + \frac{1}{(k+2)(k+4)} - \frac{1}{2(k+2)}$$

$$= \frac{3}{4} - \frac{1}{2(k+3)} + \frac{2-(k+4)}{2(k+2)(k+4)}$$

$$= \frac{3}{4} - \frac{1}{2(k+3)} - \frac{(k+2)}{2(k+2)(k+4)}$$

$$= \frac{3}{4} - \frac{1}{2(k+3)} - \frac{1}{2(k+4)}$$

$\therefore$ ~~Tru~~ $= \frac{3}{4} - \frac{1}{2[(k+1)+2]} - \frac{1}{2[(k+1)+3]}$

$\therefore$ True for $n=k+1$

Concluding statement

Q3

(ii) $$\sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)} = \lim_{n \to \infty} \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

as $n \to \infty$, $\frac{1}{2(n+2)} \to 0$

and $\frac{1}{2(n+3)} \to 0$

$$\therefore \sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)} = \underline{\underline{3/4}} \checkmark$$