Killara HIGH SCHOOL

CONSERVA PROGREDERE

STUDENT NUMBER

MATHEMATICS
Extension 1

ASSESSMENT TASK NO: 3

Date: 10th June 2011

TIME ALLOWED: *Reading time*: 5 *minutes*
Working time: 60 *minutes*

INSTRUCTIONS: This paper consists of 3 questions
ANSWER all 3 questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

HE3	uses a variety of strategies to investigate mathematical models of situations involving simple harmonic motion, or exponential growth and decay
HE5	applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
HE4	uses the relationship between functions, inverse functions and their derivatives

	MARKS
QUESTION 1 Applications of Calculus to Physical world	/17
QUESTION 2 Applications of Calculus to Physical world /Inverse Trigonometric Functions	/15
QUESTION 3 Inverse Trigonometric Functions	/14
	/46

This assessment task constitutes 20% of the Higher School Certificate Course Assessment.

Question 1 (17 marks) [BEGIN A NEW PAGE]

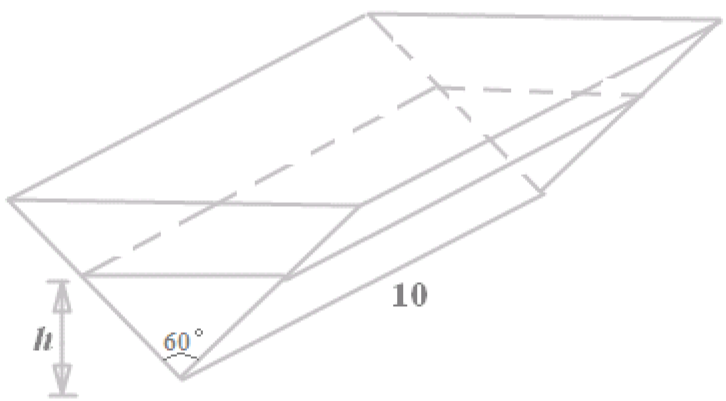
a) The velocity of a particle P is given by the equation $v = -2\sqrt{1-x^2}$, where x is the displacement of P from the origin in metres after t seconds, with movement being in a straight line.

Initially the particle is at the origin.

(i) Find an expression for the displacement of the particle in terms of t. **2**

(iii) Determine the velocity of the particle at time $\frac{\pi}{6}$ seconds. **2**

b) A large irrigation trough in the shape of a triangular prism with base angle 60° is filled with water at a constant rate of 2m^3/min. (*All lengths are in metres*).

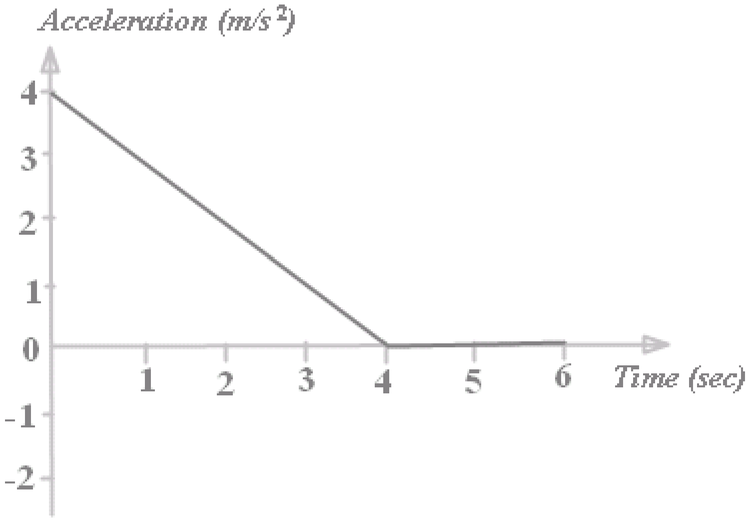


(i) Show that the equation for the volume of water in the trough in terms of h, the depth of water in metres is given by $V = \frac{10h^2}{\sqrt{3}}$. **1**

(ii) Find the rate of change of depth of water when the depth is 2 metres. **2**

Question 1 continues…

c) The graph shows the acceleration of a particle during a 6 second interval.



Initially the particle is at rest at the origin.

(i) Sketch the velocity – time graph of the particle for $0 \le t \le 6$. **3**

(ii) Calculate the distance travelled by the particle during this 6 seconds. **2**

d) A particle moves in a straight line with $\ddot{x} = 3 - 2x$. The velocity of the particle is $v = 2\ \text{ms}^{-1}$ when the displacement $x = 1$ m.

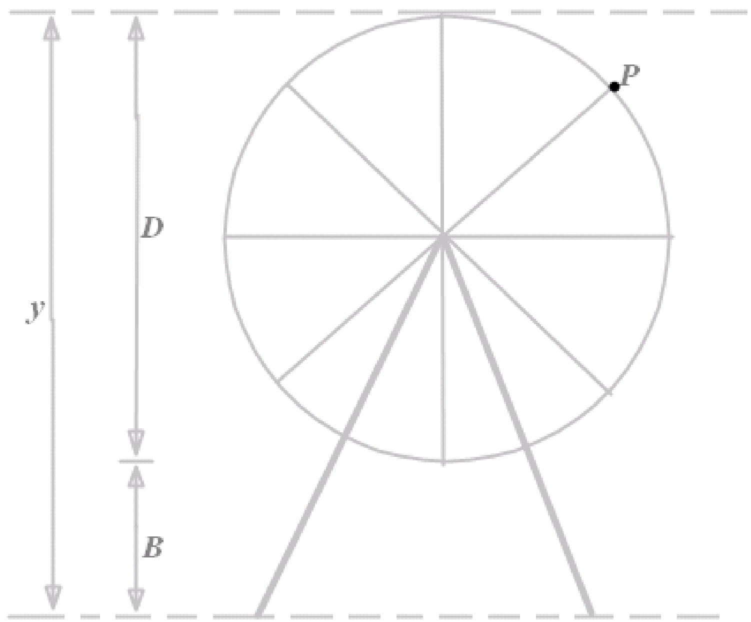
(i) Find the velocity v of the particle in terms of x. **2**

(iii) Describe the motion of the particle. **3**
(*Draw sketches of the velocity and acceleration graphs to aid your explanation.*)

End of Question 1

Question 2 (15 marks) [BEGIN A NEW PAGE]

a) The diagram shows a giant Ferris wheel at a fair ground.
B = the distance the bottom of the wheel is above the ground.
D = the diameter of the wheel
P is a point on the wheel.



The wheel rotates at a constant rate. The formula $y = 25 + 18\cos\left[\frac{\pi}{10}(t+9)\right]$ gives the height of the point P above the ground at time t seconds.

(i) Prove that the height of the point P is moving in simple harmonic motion and determine the centre of motion. **2**

(ii) Calculate the diameter of the wheel and the height the bottom of the wheel is above the ground. **2**

(iii) Determine the number of revolutions the wheel makes per minute. **2**

Question 2 continues….

Question 2 continued..

b) Evaluate:

(i) $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (in radians) **1**

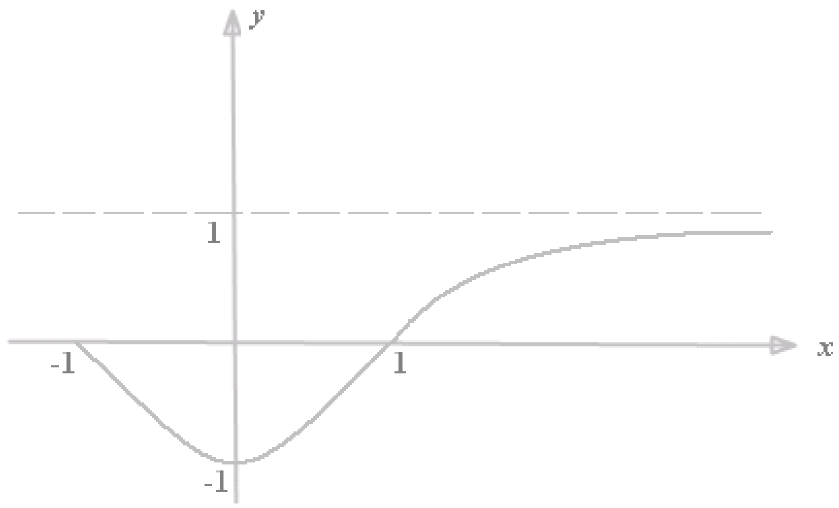
(ii) $\tan\left(\cos^{-1}\dfrac{2}{3}\right)$ (in exact form) **2**

c) Differentiate the following:

(i) $\sin^{-1}(2x+1)$ **1**

(ii) $x^2\log_e\left(\tan^{-1}x\right)$ **2**

d) For the function given below:



(i) State the domain for which $f(x)$ will have an inverse function. **1**

(ii) Sketch $y = f^{-1}(x)$, clearly showing the domain and range. **2**

End of Question 2

Question 3 (14 marks) [BEGIN A NEW PAGE]

a) Find the indefinite integral $\int \frac{dx}{\sqrt{3-x^2}}$ **1**

b) Evaluate $\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}\,dx$ **2**

c) It is given that if $f(x) = x\tan^{-1}(x) - \frac{1}{2}\log_e(x^2+1)$, then $f'(x) = \tan^{-1}(x)$

Use this information to find the area enclosed by $y = \tan^{-1}(x)$, the x-axis and the ordinates $x = -1$ and $x = 1$ (Give your answer in simplest exact form) **3**

d) Using the substitution $u = e^x$, show that

$$\int_{-1}^{1} \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1}\left(\frac{e^2-1}{2e}\right).$$ **3**

e) (i) State the domain and range of the function **2**

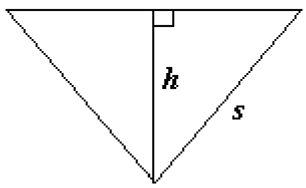
$$f(x) = \tan^{-1}(\sqrt{x}).$$

(i) Find $f'(x)$ and comment on the behaviour of $f'(x)$, for $x \geq 0$. **2**

(iii) Hence, draw a sketch of $y = f(x)$. **1**

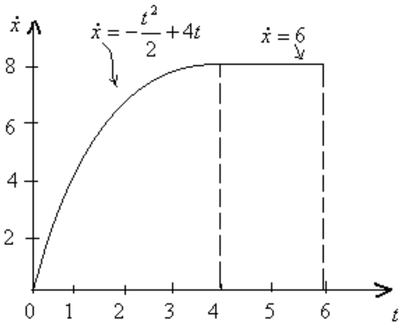
End of paper

Year 12 Extension 1 Task 3: Marking Criteria

a)(i) $v = -2\sqrt{1-x^2}$ $t = 0, x = 0$ $\frac{dx}{dt} = -2\sqrt{1-x^2}$ $\therefore \frac{dx}{\sqrt{1-x^2}} = -2dt$ $\text{Sin}^{-1}x = -2t + C$ $t = 0, x = 0 \Rightarrow C = 0$ $\therefore \sin^{-1}x = -2t$ $\therefore x = \sin(-2t)$ $x = -\sin(2t)$ (ii) when $t = \frac{\pi}{6}$, $x = -\frac{\sqrt{3}}{2}$ $v = -2\sqrt{1-x^2}$ $= -2\sqrt{1-\frac{3}{4}}$ $= -1 \text{ ms}^{-1}$.	**1 mark for expressing v as $\frac{dx}{dt}$ and getting the correct integral $\text{Sin}^{-1}x = -2t + C$** **1 mark for evaluating C as getting the correct expression for x.** **1 mark for substituting into the expression for x in (i) and evaluating x.** **1 mark for correct substitution to v and**	*Many students forgot to make x subject of the formula.*
(b)  (i) $S = \frac{h}{\sin 60°} = \frac{2h}{\sqrt{3}}$ Volume of the trough = $\frac{1}{2} \times sh \times 10$ $= \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \times 10$		*"Show that" question must have every line of verbal and numerical working out.*

$V = \dfrac{10h^2}{\sqrt{3}}$ (ii) $\dfrac{dv}{dt} = 2\text{m}^3 / \text{min}$ $\dfrac{dv}{dt} = \dfrac{dv}{dh} \times \dfrac{dh}{dt}$ $V = \dfrac{10h^2}{\sqrt{3}}$ $\therefore \dfrac{dv}{dh} = \dfrac{10}{\sqrt{3}} \times 2h = \dfrac{20h}{\sqrt{3}}$ $\dfrac{dh}{dt} = \dfrac{dh}{dv} \times \dfrac{dv}{dt}$ $= \dfrac{\sqrt{3}}{20h} \times 2$ When $h = 2$, the rate of change of depth is $\dfrac{\sqrt{3}}{20}$ m/min.	**1 mark: for the correct proof.** **(Every step of the work must be shown)** 1 mark for $\dfrac{dv}{dh} = \dfrac{20h}{\sqrt{3}}$ **1 mark: for correct solution**	
c) (i) For $0 < t < 4$, $\ddot{x}_1 = -t + 4$ $\therefore$ velocity $\dot{x}_1 = -\dfrac{t^2}{2} + 4t + C$ When $t = 0$, $\dot{x}_1 = 0$ $\therefore C = 0$ $\therefore \dot{x}_1 = -\dfrac{t^2}{2} + 4t$ For $4 < t < 6$, $\ddot{x}_2 = 0$, $\dot{x}_2 = \int 0dt = C_1$ When $t = 4$, Velocity $= -\dfrac{4^2}{2} + 4 \times 4 = 8$ $\therefore C_1 = 8.$ $\therefore \dot{x}_2 = 8$	**1 mark: for developing** $\dot{x}_1 = -\dfrac{t^2}{2} + 4t$ **1 mark: for finding for the velocity expression** $\dot{x}_2 = 8$	*For straight line graphs y = f(x), to sketch the graph of* $\int f(x)dx$ *You must work out the exact equations. In this question, there was adequate information available to do this.*

<table>
<tr>
<td>



(ii) Distance travelled = Area under the velocity curve.

$D = A_1 + A_2$

$= \int_0^4 -\frac{t^2}{2} + 4t\,dt \; + \; 2 \times 8$

$= \left[-\frac{t^3}{6} + 2t^2 \right]_0^4 \; + 16$

$= 21\frac{1}{3} \; + 16 \; = \; 37\frac{1}{3}$ metres

</td>
<td>

1 mark: For the graph

1 mark: Evaluates area A_1 (area under the parabola).

1 mark: Evaluates area A_2 = 16 sq. units and adds it to A_1.

</td>
<td>

Many students did not get this graph correct. Even those who got it, did not have sufficient information on it. You absolutely needed to have y-intercept 8. Y-intercept of 4 was very common. Students thought that the y-intercept of $\dot{x}$ and $\ddot{x}$ are exactly the same.

</td>
</tr>
<tr>
<td>

d) (i) $\ddot{x} = 3 - 2x$.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 2x$.

Integrating both sides w.r.to x,

$\frac{1}{2}v^2 = 3x - x^2 + C$

$x = 1, v = 2 \;\; \therefore \; C = 0$

$v^2 = 6x - 2x^2$

$v^2 = 2x\,(3 - x)$

$x = 1, v = 2$

$\therefore \quad v = \sqrt{2x(3-x)}$

</td>
<td>

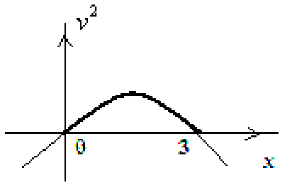
1 mark: sets $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and integrates w.r.to x to get

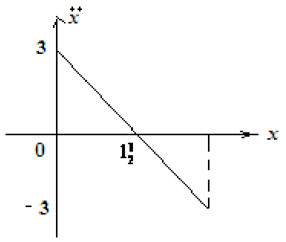
$\frac{1}{2}v^2 = 3x - x^2 + C$

1 mark: Evaluates C and gets the expression $v = \sqrt{2x(3-x)}$

Must explain why only the positive root was chosen.

</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</table>

<table>
<tr>
<td>
(ii)




$v^2 \geq 0 \;\; \therefore \;\; 0 \leq x \leq 3.$

$\ddot{x} = 3 - 2x$

And $v = \sqrt{2x(3-x)}$

At $x = 1$, $v = 2$ means that the particle will move right speeding up ($v > 0$ and $\ddot{x}$ > 0) attains maximum speed at $x = 1\frac{1}{2}$ ($\ddot{x}$=0) .
For $x > 1\frac{1}{2}$, $v > 0$ and $\ddot{x}$<0, $\therefore$ the particle slows down until it comes to rest at $x = 3$. But the force ($\ddot{x}$) acting on it is negative, $\therefore$ the particle moves to the origin. Ie. The particle oscillates between 0 and 3.
$\ddot{x} = 3 - 2x$

$= -2\left(x - \frac{3}{2}\right)$ is of the form $-n^2(x - b)$.

$\therefore$ the motion is SHM.

</td>
<td>
1 mark for point

1 mark for point 2

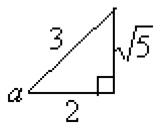
1 mark for point 3
</td>
<td>
Very poorly done.

When describing motion, you must comment as x increases, how are $\dot{x}$ and $\ddot{x}$ are affecting the motion.

Follow the particle (make the comments progressively) to describe motion; not picking points here and there and giving fragmented description.
</td>
</tr>
</table>

Question 2

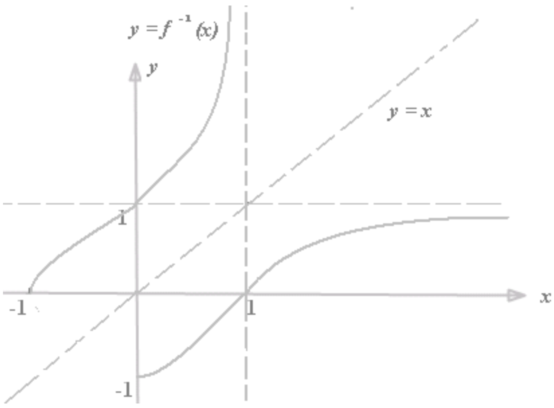
(a) (i) $y = 25 + 18\cos\left[\frac{\pi}{10}(t+9)\right]$ $\dot{y} = -18\sin\left[\frac{\pi}{10}(t+9)\right] \times \frac{\pi}{10}$ $\ddot{y} = -18\left(\frac{\pi}{10}\right)^2 \cos\left[\frac{\pi}{10}(t+9)\right]$ $= -18\left(\frac{\pi}{10}\right)^2\left(\frac{y-25}{18}\right)$ $= -\left(\frac{\pi}{10}\right)^2(y-25)$ This is SHM about $y = 25$.	**1 mark: for correct velocity expression** **1 mark: manipulating to get $\ddot{y} = -n^2(x-b)$ and hence SHM.**	
(ii) $-1 \le \cos\left[\frac{\pi}{10}(t+9)\right] \le 1$ $\therefore\ y_{\max} = 25 + 18 = 43$ $y_{\min} = 25 - 18 = 7$ $\therefore$ D = 36 B = 7	**1 mark: for correct D** **1 mark: for correct B** **(Reduce 1 mark if $-1 \le \cos\left[\frac{\pi}{10}(t+9)\right] \le 1$ is not given)**	
(iii) Period = $\frac{2\pi}{\frac{\pi}{10}}$ = 20 sec $\therefore$ No: of revolutions/min = $\frac{60}{20}$ = 3	**1 mark: for correct period** **1 mark: correct no: of revolutions using the period from (ii)**	

b))(i) $\frac{\pi}{3}$ (ii)) Let a = $\cos^{-1}\frac{2}{3}$ then, cosa = $\frac{2}{3}$  tan a = $\frac{\sqrt{5}}{2}$ $\therefore \tan(\cos^{-1}\frac{2}{3}) = \frac{\sqrt{5}}{2}$	**1 mark for correct answer** **1 mark for the value of cos*a* and the representation with a diagram** **1 mark for correct tan*a* using the diagram.**	

c)(i) $\frac{d}{dx}\left[\sin^{-1}(2x+1)\right]$ $\frac{1\times 2}{\sqrt{1-(2x+1)^2}}$ (ii) $\frac{d}{dx}\left[x^2 \log_e(\tan^{-1}x)\right]$ u = x^2 u′ = $2x$ v = $\log_e(\tan^{-1}x)$ v′ = $\frac{1}{\tan^{-1}x(1+x^2)}$ $\therefore \frac{d}{dx}\left[x^2\log_e(\tan^{-1}x)\right] =$ 2x × $\left[\log_e(\tan^{-1}x)\right] + \frac{x^2}{\tan^{-1}x(1+x^2)}$	**1 mark** **1 mark for correct differentiation of u and v.** **1 mark for applying product rule correctly.**	

<table>
<tr>
<td>d) (i) Largest possible domain

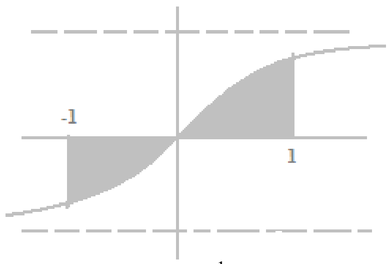
D: $x \geq 0$

(ii)


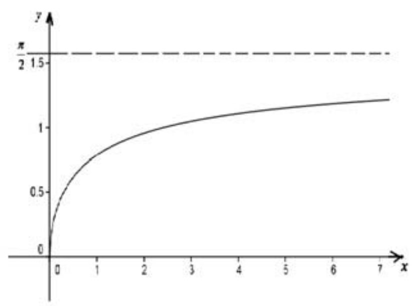
1 mark: correct domain and range of $y = f^{-1}(x)$

1 mark: correct shape of $y = f^{-1}(x)$ including the correct asymptote</td>
<td>1 mark</td>
<td></td>
</tr>
</table>

Question 3

a) $\int \frac{dx}{\sqrt{3-x^2}} =$ $\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)$+C	**1 mark for correct answer**	*Well done except that some students forgot to $\sqrt{3}$ and TSI*
b) $\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}dx$ $= \int_0^{\frac{2}{3}} \frac{1}{9\left(\frac{4}{9}-x^2\right)}dx$ $= \frac{1}{9} \times \frac{3}{2}\left[\tan^{-1}\left(\frac{3}{2}x\right)\right]_0^{\frac{2}{3}}$ $= \frac{1}{6}\{\tan^{-1}(1) - \tan^{-1}(0)\}$ $= \frac{1}{6} \times \frac{\pi}{4} = \frac{\pi}{24}$	**1 mark for correct integration** **1 mark for correct evaluation**	*Reasonably well done. The formula is on the table of standard integrals.*
c)  Required area = $\int_{-1}^{1} ydx$ $= \int_{-1}^{1} \tan^{-1}(x)dx$ $= 2\times \int_0^1 \tan^{-1}(x)dx$ $= \frac{1}{2}\left[x\tan^{-1}(x) - \log_e(x^2+1)\right]_0^1$ $= \frac{1}{2}\left[\left(\frac{\pi}{4} - \ln(2)\right) - (0 - \ln 1)\right]$ $= \frac{\pi}{8} - \frac{1}{2}\ln(2)$	**1 mark for understanding it is an odd function and therefore multiplying by 2** **1 mark for using the correct integral from the data** **1 mark for correct evaluation**	*Students who took the time to draw a diagram did this well. They quickly realised that they could double the integral from 0 to 1. Students who tried to take the absolute value of the interval form -1 to o got into a mess.* *Students who struggled to attempt this question are reminded that a quick diagram can prompt thought processes* *Students who chose to ignore the data and take the area between the curve and the y axis were unsuccessful and got into a mess.* *USE THE INFORMATION*

d) $\displaystyle\int_{-1}^{1} \frac{e^x}{1+e^{2x}}\,dx$ Let u = e^x $\therefore du = e^x dx$ $x=-1, \quad u=\frac{1}{e}$ $x=1, \quad u=e$	**1 mark for correct conversions**	*Students who changed the terminals in the start of the question were more successful.*
$\displaystyle\int_{\frac{1}{e}}^{e} \frac{du}{1+u^2}$ $= \left[\tan^{-1}(u)\right]_{\frac{1}{e}}^{e}$ $= \tan^{-1}(e) - \tan^{-1}\left(\frac{1}{e}\right)$	**1 mark for correct integration**	*The first part of the question was well done.* *Students are reminded that SHOW means you need to show every step of working.*
Let $k = \tan^{-1}(e) - \tan^{-1}\left(\frac{1}{e}\right)$ $\therefore$ *tan k* = $\tan\left(\tan^{-1}(e) - \tan^{-1}\left(\frac{1}{e}\right)\right)$ $= \dfrac{e-\frac{1}{e}}{1+e\times\frac{1}{e}}$ $= \left(\dfrac{e^2-1}{2e}\right)$ $\therefore k = \tan^{-1}\left(\dfrac{e^2-1}{2e}\right).$	**1 mark for the correct solution**	*You cannot factorise inverse trig functions nor use a mutation of log laws for inverse trig* *The third part of this question was not done well.*

e) $f(x) = \tan^{-1}\left(\sqrt{x}\right)$ (i) Domain: $x \geq 0$ **1 mark** Range: $0 \leq y < \frac{\pi}{2}$ **1 mark** (ii) $f'(x) = \frac{1}{1+x} . \frac{1}{2\sqrt{x}}$ $= \frac{1}{2\sqrt{x}(1+x)}$, $x \neq 0$ **1 mark** Since, $x > 0$, $f'(x) > 0$, hence, $f(x)$ is increasing from $x = 0$ to $\frac{\pi}{2}$, $y \neq \frac{\pi}{2}$. **1 mark** (iii) $f'(x)$ undefined at $x = 0$ and $f'(x) \to \infty$, as $x \to \infty$. 	(i) **1 mark: for correct domain** **1 mark: for correct range** **(ii)** **1 mark for $f'(x)$** **1 mark stating $f'(x) > 0$ in the domain given and determines corresponding $f(x)$ values. I gave 1 mark for $f'(x) > 0$ and $x \neq 0$** **(iii)** **Sketch 1 mark.** **Must show vertical gradient at $x = 0$ and asymptote.**	*(i)Students are reminded to always notice square roots and denominators!! Many students could find the domain but failed to see the effect of a change in domain on the range. Many students forgot that there is an asymptote on pi/2 so wrote the domain as* $\leq \pi$ *(ii) many students could differentiate this correctly and obtained 1 mark, but some forgot to differentiate the* $\sqrt{x}$. *The second part of this was very poorly done. Many students commented on f(x). Again, square roots and denominators!!* *(iii) One students obtain the 1 mark 4 this Q. Students seem to have no idea what an undefined gradient means. We need to get them to find the gradient of say, y=4. Many students were able to obtain the correct shape of the graph and the asymptote.*