Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS EXTENSION 1

HSC ASSESSMENT TASK 2

15th March, 2012

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 2 hours

INSTRUCTIONS: This paper consists of 10 multiple choices and 4 fifteen mark questions
ANSWER all questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

H1 - seeks to apply mathematical techniques to problems in a wide range of practical contexts
H2 - constructs arguments to prove and justify results
H4 - expresses practical problems in mathematical terms based on simple given models
H5 - applies appropriate techniques from the study of calculus and trigonometry to solve problems
H6 - uses the derivative to determine the features of the graph of a function
H7 - uses the features of a graph to deduce information about the derivative
H9 - communicates using mathematical language, notation, diagrams and graphs
Plus Preliminary from P1 to P8
HE1 - Appreciates interrelationships between ideas drawn from different areas of mathematics
HE6 – Determines integrals by reduction to a standard form through a given substitution

This assessment task constitutes 25% of the Higher School Certificate Course Assessment.

Multiple choices (1 mark each)
Use the answer sheet for Q1 to 10

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1. *If* $f(x)=\begin{cases} 8x^3 & if\ x>3 \\ 3x^2-2 & if\ 0\le x\le 3 \\ 9 & if\ x<0 \end{cases}$ $\qquad f(3)+f(1)+f(-1)=\ldots\ldots$

(A) 35 *(B) 226* *(C) 233* *(D) 53*

2. $\lim\limits_{\theta\to 0}\dfrac{\tan\frac{\theta}{6}}{\theta}=\ldots\ldots\ldots\ldots$

(A) 6 (B) $\frac{1}{6}$ (C) 1 (D) 3

3. The coordinate of the focus of the parabola $x^2 - 2x + 2y - 11 = 0$ is:

(A) (1, 5 ½) (B) (1, 6 ½) (C) -1, -6) (D) (2, 3)

4. If $\tan\theta = \sqrt{3}$ and $\cos\theta < 0$, then $\sin\theta = \ldots$

(A) 1 (B) ½ (C) $\frac{\sqrt{3}}{2}$ (D) $-\frac{\sqrt{3}}{2}$

5. Consider the function $f(x)=\dfrac{x^2-1}{x-4}$,

As $x\to\infty$, $f(x)\to$?

(A) 0 (B) ¼ (C) x - 1 (D) x

6. Which of the following is true?

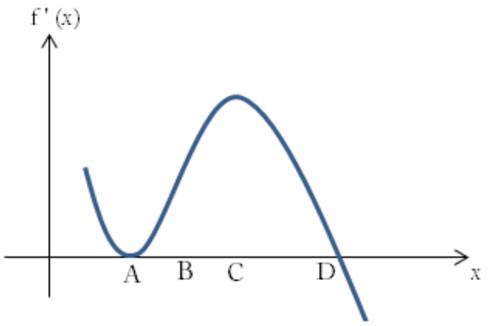
(A) $\log_{10}\left(\frac{x^2}{y}\right) = \frac{2\log_{10} x}{\log_{10} y}$

(B) $\log_2 16 + \log_2 8 = 12$

(C) $e^{\ln x} = \ln x$

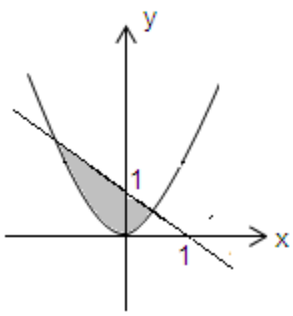
(D) $\log_3 \frac{1}{81} = -4$

7. Below is the graph of $f'(x)$. $f(x)$ has the horizontal point of inflexion at :



(A) A (B) B (C) C (D) D

8. The inequalities which correspond to the region shaded is:



(A) $y \leq 1-x$ *and* $y \leq x^2$

(B) $y < 1-x$ *and* $y \geq x^2$

(C) $y \leq 1-x$ *and* $y \geq x^2$

(D) $y \leq 1+x$ *and* $y \geq x^2$

9. Consider the function $f(x)=\dfrac{x^2-2x-3}{(x+1)(x-4)}$. Which of the following statement is true?

(A) $f(x)$ has horizontal asymptotes at x = -1 and x = 4

(B) $f(x)$ has vertical asymptotes at x = -1 and x = 4

(C) $f(x)$ is discontinuous at x = -1

(D) None of the above.

10. The polynomial $P(x) = x^3 + px^2 + qx + r$ has real roots $\sqrt{\beta}$, - $\sqrt{\beta}$ and α. Find the value for r in term of p and q.

(A) $r=-\dfrac{p}{q}$ (B) $r=\dfrac{p}{q}$ (C) $r=-pq$ (D) $r = pq$

Total marks − 60

Attempt all Questions
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available

Question 11 *(15 marks) Use a* ***SEPARATE*** *writing booklet* *Mark*

a) Prove that $\dfrac{1}{\sqrt{5}+1}+\dfrac{1}{\sqrt{5}+3}$ is rational. 2

b) Differentiate $e^{2x}(\cos x - \sin x)$ 2

c) Solve the inequality $\dfrac{x-1}{2x-1} \leq 3$ 3

d) Simplify $\dfrac{4^{n-1} \times 2^{n}}{8^{n-3}}$ 2

e) The acute angle between the lines $y = 3x + 5$ and $y = mx + 4$ is $45°$. Find the two possible values of m. 3

f) i) Sketch the graph of $y = |3 - 2x|$ 2

ii) For what values of b does $|3-2x| = b - x$ have two solutions? 1

***Question 12** (15 marks) Use a **SEPARATE** writing booklet* *Mark*

a) Find the indefinite integral: $\int \sqrt{x}\left(1+\frac{1}{x}\right)dx$ 1

b) Find the exact answer of $\int_{\sqrt{2}}^{2} \frac{4x+2}{x^2+x+1}\,dx$ 2

c) i) Determine whether y = $\sin x.\tan^2 x$ is an odd or even function. 2

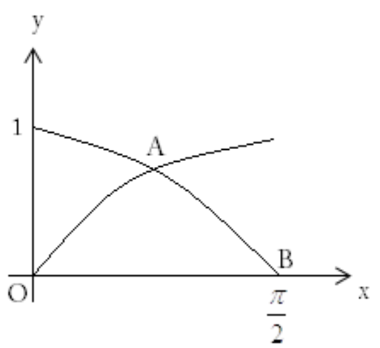
ii) Hence, otherwise evaluate $\int_{-1}^{1} \sin x \tan^2 x dx.$ 1

d) Using the substitution $u = 2x^2 + 1$ to find $\int x(2x^2+1)^{\frac{5}{3}}\,dx$ 2

e) Show that $\frac{1+\sin^2 x}{\cos^2 x} = 2\sec^2 x - 1.$ 2

Hence, otherwise evaluate $\int_{0}^{\frac{\pi}{3}} \frac{1+\sin^2 x}{\cos^2 x}\,dx$ 2

f) 3



NOT DRAWN TO SCALE

The diagram shows portions of the graphs y = sinx and y = cosx. Fin the area of the region enclosed by the arc OA, the arc AB and the interval OB.

Question 13 (15 marks) *Use a **SEPARATE** writing booklet* *Mark*

a) Let $f(x) = x + \log_e x$.

(i) Write down the natural domain for $f(x)$. 1

(ii) Show that for all values of x in the natural domain, $y = f(x)$ is increasing 1

(iii) Show that the curve $y = f(x)$ cuts the x axis between x = 0.5 and x = 1. 1

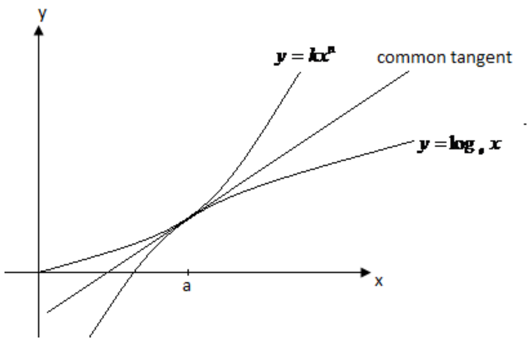
(iv) Use Newton's method with a first approximation of x = 0.5 to find a second approximation to the root of $x + \log_e x = 0$. 2

b) Find the exact volume of the solid formed when the curve $y = \log_e(2x)$ is rotated about the y-axis from y = 1 to y = 3. 3

c) i) Show that $\dfrac{d}{dx}\ln\left(\dfrac{x-1}{x+1}\right) = \dfrac{2}{(x-1)(x+1)}$ 2

ii) Hence find the exact answer of $\displaystyle\int_2^5 \frac{2}{(x-1)(x+1)}\,dx$ 1

d)



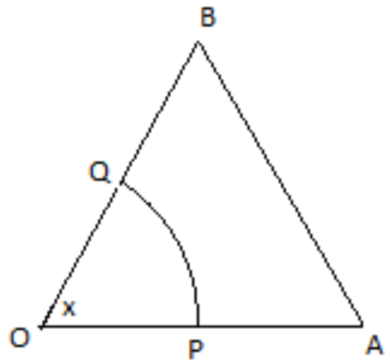
The graphs of the functions $y = kx^n$ and $y = \log_e x$ have a common tangent at $x = a$, as shown in the diagram.

i) Show that $a^n = \dfrac{1}{nk}$ 2

ii) Express k as a function of n by eliminating a. 2

Question 14 (15 marks) *Use a **SEPARATE** writing booklet* *Mark*

a)



In ΔOAB, $OA = OB = a$, which is a constant, $\angle AOB = x$ radians, where x is variable. PQ is a circular arc, centre O and radius r. If the area of ΔOAB is twice that of sector OPQ,

i) Express r^2 in terms of a and x 2

ii) Find r when $\angle AOB$ is a right angle 1

iii) Describe the behavior of r as $x \to 0$ 1

b) Prove by mathematical induction, for all odd positive integers n, $2^n + 7^n$ is divisible by 9. 4

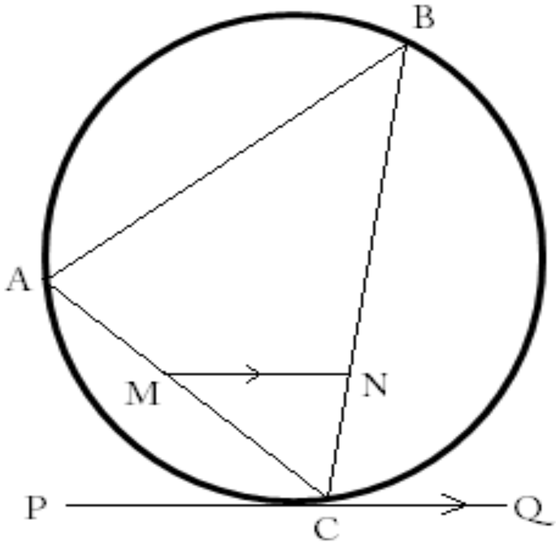
c) By equating the coefficients of sin x and cos x, or otherwise, find constants A and B satisfying the identity

i) $A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x$ 2

ii) Hence evaluate $\int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\,dx$ 1`

Question 14 continue over page

d)



Copy or trace the diagram into your writing booklet.

PCQ is the tangent at C; PCQ // MN.

i) Prove ABNM is a cyclic quadrilateral. 2

ii) BM and AN, both produced meet the circle again at X and Y. 2
Prove XY// PCQ.

END OF PAPER

2012- EX1 Y12- Half Yearly- Solution

1. 27 – 2 + 3 -2 + 9 = 35 **A**

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2. $\lim\limits_{\theta \to 0} \dfrac{\tan \frac{\theta}{6}}{\theta} = \dfrac{1}{6} \lim\limits_{\theta \to 0} \dfrac{\tan \frac{\theta}{6}}{\frac{\theta}{6}} = \dfrac{1}{6}$ **B**

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3. $x^2 - 2x + 2y - 11 = 0$

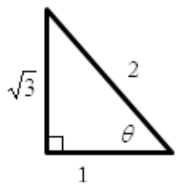
$x^2 - 2x + 1 = -2y + 12$

$(x - 1)^2 = -2(y - 6)$

$\Rightarrow$ a = ½

$\therefore$ S (1, 5 ½) **A**

4.



D

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5. $\lim\limits_{x \to \infty} \dfrac{x^2 - 1}{x - 4} = \lim\limits_{x \to \infty} \dfrac{\frac{x^2}{x} - \frac{1}{x}}{\frac{x}{x} - \frac{4}{x}} = x$ **D**

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6. $\log_3 \dfrac{1}{81} = \log_3 3^{-4} = -4$ **D**

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7. At A $f(x) = f'(x) = 0$ **A**

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8. C

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9. $f(x) = \dfrac{x^2 - 2x - 3}{(x+1)(x-4)} = \dfrac{\cancel{(x+1)}(x-3)}{\cancel{(x+1)}(x-4)}$

$f(x)$ has a vertical asymptote at x = 4

$f(x)$ is discontinuous at x = - 1 **C**

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10. P(x) = $x^3 + px^2 + qx + r$

$\sum \alpha = \sqrt{\beta} - \sqrt{\beta} + \alpha$

$= \alpha$ **(1)**

$= -p$

$\sum \alpha\beta = \sqrt{\beta} \times -\sqrt{\beta} + \alpha\sqrt{\beta} - \alpha\sqrt{\beta}$

$= -\beta$ **(2)**

$= q$

$\sum \alpha\beta\gamma = -\sqrt{\beta}.\sqrt{\beta}.\alpha$

$= -\alpha\beta$ **(3)**

$= -r$

Sub. **(1)** & **(2)** into **(3)**

$\Rightarrow$ $-p \times (-q) = r$

$\therefore$ r = pq **D**

Question 11 **Mark**

(a) $\dfrac{1}{\sqrt{5}+1}+\dfrac{1}{\sqrt{5}+3}=\dfrac{\sqrt{5}-1}{4}-\dfrac{\sqrt{5-3}}{4}$ **1**

$=\dfrac{1}{2}$ is rational **1**

(b) $y' = 2e^{2x}(\cos x - \sin x) + e^{2x}(-\sin x - \cos x)$ **1**

$= e^{2x}(2\cos x - 2\sin x - \cos x - \sin x)$

$= e^{2x}(\cos x - 3\sin x)$ **1**

(c) $\dfrac{x-1}{2x-1} \le 3$ $\qquad x \ne \dfrac{1}{2}$

$(x-1)(2x-1) \le 3(2x-1)^2$ **1**

$(2x-1)(3(2x-1) - x + 1) \ge 0$

$(2x-1)(5x-2) \ge 0$ **1**

$\therefore x > \frac{1}{2}$ and $x \le \dfrac{2}{5}$ **1**

(d) $\dfrac{4^{n-1}\times 2^n}{8^{n-3}}=\dfrac{2^{2n-2}\times 2^n}{2^{3n-9}}$ **1**

$= 2^7$ **1**

(e) $y = 3x + 5$, $\qquad y = mx + 4$

$\tan 45^\circ = \left|\dfrac{3-m}{1+3m}\right|$ **1**

$= 1$

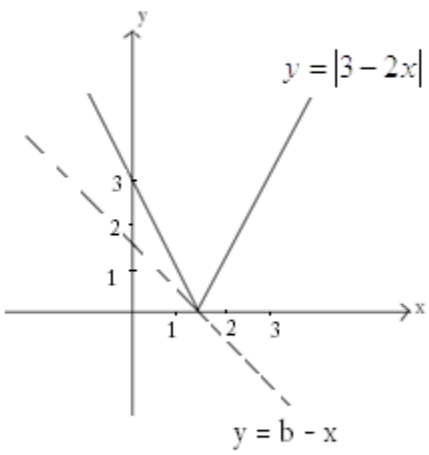
$\dfrac{3-m}{1+3m} = 1 \quad or \quad \dfrac{3-m}{1+3m} = -1$

$\Rightarrow \quad 3-m = 1+3m \qquad 3-m = -1-3m$

$4m = 2 \qquad 2m = -4$ **1 mark for each m**

$m = \dfrac{1}{2} \quad or \quad m = -2$

- 1 (for any mistake, incorrect scale, or no labelling of axes)



As the line y = b – x moves up from the current position,
it will cut the graph $y = |3-2x|$ at two points.

∴ b > 1 ½ **1**

Question 12

(a) $\int \sqrt{x}\,(1+\frac{1}{x})\,dx = \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ **1**

(b) $\int_{\sqrt{2}}^{2} \frac{4x+2}{x^2+x+1}\,dx = \left[2\ln(x^2+x+1)\right]_{\sqrt{2}}^{2}$ **1**

$= 2(\ln 7 - \ln(3+\sqrt{2}))$

$= 2\ln\left(\frac{7}{3+\sqrt{2}}\right)$ **1**

(correct simplified answer)

(c) i) f(x) = sinx . tan ^{2} x

f(-x) = sin(-x) . tan ^{2} (-x) **1 (substitution of –x)**

= - sin x. (- tan x) ^{2}

= - sin x. tan ^{2} x **1 (for correct following steps, no fudging)**

= - f(x)

∴ f(x) is an odd function. **- 1 (for no conclusion)**

Hence $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin x.\tan^2 x\,dx = 0$ **1**

(d) $\int x(2x^2+1)^{\frac{5}{3}}\,dx$ let u = 2x ^{2} + 1, du = 4x dx

$= \frac{1}{4}\int u^{\frac{5}{3}}\,du$ **1 (correct changing variables)**

$= \frac{1}{4}\times\frac{3}{8}\times u^{\frac{8}{3}} = \frac{3}{32}u^{\frac{8}{3}}$

∴ $\int x(2x^2+1)\,dx = \frac{3}{32}(2x^2+1)^{\frac{8}{3}} + C$ **1 (correct answer)**

Question 12 (cont.)

(e) $\dfrac{1+\sin^2 x}{\cos^2 x} = 2\sec^2 x - 1$

$$\text{L.H.S} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \qquad \mathbf{1}$$

$$= \sec^2 x + \tan^2 x$$

$$= \sec^2 x + \sec^2 x - 1$$

$$= 2\sec^2 x - 1 \qquad \mathbf{1}$$

$$\int_0^{\frac{\pi}{3}} \frac{1+\sin^2 x}{\cos^2 x}\,dx = \int_0^{\frac{\pi}{3}} 2\sec^2 x - 1\,dx$$

$$= \left[2\tan x - x\right]_0^{\frac{\pi}{3}} \qquad \mathbf{1\ (correct\ integration)}$$

$$= 2\sqrt{3} - \frac{\pi}{3} \qquad \mathbf{1\ (correct\ answer)}$$

(f) At A, sin x = cos x $\Rightarrow$ x = $\dfrac{\pi}{4}$

$$A = \int_0^{\frac{\pi}{4}} \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx \qquad \mathbf{1}$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{4}} + \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \mathbf{1}$$

$$= 2 - \frac{2}{\sqrt{2}} \quad \text{or} \quad \frac{4-\sqrt{2}}{2} \qquad \mathbf{1}$$

Question 13

(a) i) Domain : $x > 0$ **1**

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ii) $f'(x) = 1 + \dfrac{1}{x} > 0$ for all $x > 0$ **1**

$\therefore$ f(x) is increasing for all x in the natural domain

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iii) $f(0.5) = 0.5 + \log_e 0.5$

$= -0.1935 < 0$ **1 (for both correct steps plus conclusion)**

$f(1) = 1 + \ln 1$

$= 1 > 0$

$\therefore$ The curve cuts the x-axis between x = 0.5 and x = 1

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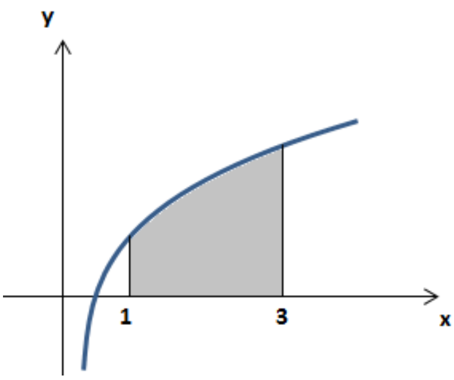
iv) $f'(0.5) = 1 + \dfrac{1}{0.5}$ **1**

$= 3$

$a_1 = 0.5 - \dfrac{-0.1931}{3}$

$= 0.564$ **1**

(b)



$y = \log_e 2x$

$\Rightarrow x = \dfrac{e^y}{2}$

$\therefore x^2 = \frac{1}{4} e^{2y}$ **1 (correct changing variables)**

$$V = \frac{\pi}{4}\int_1^3 e^{2y}\, dx = \frac{\pi}{4} \times \frac{1}{2}\left[e^{2y}\right]_1^3$$ **1**

$$= \frac{\pi}{8}(e^6 - e^2) \text{ unit}^3$$ **1**

Question 13 (cont.)

(c) i) $\ln\left(\dfrac{x-1}{x+1}\right) = \ln(x-1) - \ln(x+1)$

$$\frac{d}{dx}\ln\left(\frac{x-1}{x+1}\right) = \frac{d}{dx}\left[\ln(x-1) - \ln(x+1)\right] \qquad \mathbf{1}$$

$$= \frac{1}{x-1} - \frac{1}{x+1}$$

$$= \frac{2}{(x-1)(x+1)} \qquad \mathbf{1}$$

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ii) $$\int_2^5 \frac{2}{(x-1)(x+1)}\,dx = 2\left[\ln\left(\frac{x-1}{x+1}\right)\right]_2^5$$

$$= 2\left(\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\right)\right) = 2\ln 2 \quad \text{or} \quad \ln 4 \qquad \mathbf{1}$$

(d) i) $y_1 = kx^n \Rightarrow y_1' = nkx^{n-1}$

$y_2 = \log_e x \Rightarrow y_2' = \dfrac{1}{x}$

Both functions have the same tangent at x = a.

$\therefore$ $m_1 = m_2$ i.e. $nkx^{n-1} = \dfrac{1}{x}$, Sub. x = a **1 (showing $m_1 = m_2$)**

$\Rightarrow nka^{n-1} = \dfrac{1}{a}$ $\quad\therefore\quad a^n = \dfrac{1}{nk}$ (1) **1**

also $a = \dfrac{1}{\sqrt[n]{nk}}$ (2)

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ii) y_1 and y_2 have the point of intersection at x = a

$\therefore$ $kx^n = \log_e x$

At x = a, $ka^n = \log_e a$ Sub. (1) & (2) **1 (showing $y_1 = y_2$ at x = a)**

$$\Rightarrow \quad k \times \frac{1}{nk} = \log_e (nk)^{-\frac{1}{n}}$$

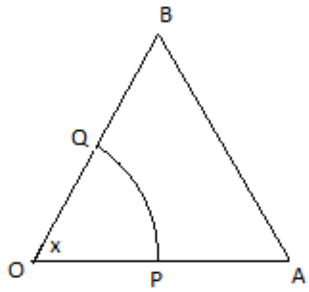
$$\frac{1}{n} = -\frac{1}{n} \log_e (nk)$$

$$\Rightarrow \; e^{-1} = nk \qquad \mathbf{1}$$

$$\therefore \quad k = \frac{1}{en}$$

Question 14

(a)



i)

$$A_{\Delta OAB} = \frac{1}{2} a^2 \sin x, \qquad A_{\sec tor\, OPQ} = \frac{1}{2} r^2 x$$

1 (correct finding both areas)

Given $A_{\Delta OAB} = 2\, A_{\sec tor\; OPQ}$

$$\therefore \quad \frac{1}{2} r^2 \sin x = r^2 x$$

$$\Rightarrow \quad r^2 = \frac{a^2 \sin x}{2x}$$

1

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ii) when $x = \dfrac{\pi}{2}$, $\quad r^2 = \dfrac{a^2}{\pi} \quad \Rightarrow \quad r = \dfrac{a}{\sqrt{\pi}}$ **1**

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iii)

$$\lim_{x \to 0} \frac{a^2 \sin x}{2x} = \frac{a^2}{2} \lim_{x \to 0} \frac{\sin x}{x}$$

$$= \frac{a^2}{2}$$

1

(-1 for missing or incorrect conclusion)

(b) i) Prove it is true for n = 1 **1**

$2^1 + 7^1 = 9$ is divisible by 9

$\therefore$ it is true for n = 1.

Assume it is true for n = k

i.e $2^k + 7^k = 9P$, where P is an integer.

$\Rightarrow \quad 7^k = 9P - 2^k \qquad (1)$

Prove it is true for n = k + 2 (odd integer).

i.e. to prove: $2^{k+2} + 7^{k+2} = 9Q$, where Q is an integer. **1**

L.H.S = $4 \times 2^k + 49 \times 7^k$

= $4 \times 2^k + 49 \times (9P - 2^k)$ Using (1) **1**

= $9 \times 49P + 4 \times 2^k - 49 \times 2^k$

= $9 \times 49P + 45 \times 2^k$

= $9\,(49\,P + 5 \times 2^k)$

= 9Q where $Q = 49\,P + 5 \times 2^k$ **1**

$\therefore$ It is true for n = k + 2.

The formula is true for n = 1, true for n = 2 (from step 3), n = 3 nd so on for any odd integer. Hence by mathematical induction $2^n + 7^n$ is divisible by 9 for all odd positive integers.

(c) i) $A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x$.

L.H.S = $2A\sin x + A\cos x + 2B\cos x - B\sin x$

$= (A + 2B)\cos x + (2A - B)\sin x$

$\equiv \sin x + 8\cos x$.

$\therefore$ $A + 2B = 8$

$2A - B = 1 \quad (\times 2) \qquad \Rightarrow \qquad 4A - 2B = 2$

$+$

$A + 2B = 8$

$5A = 10$

$\therefore$ A = 2 and B = 3 **1**

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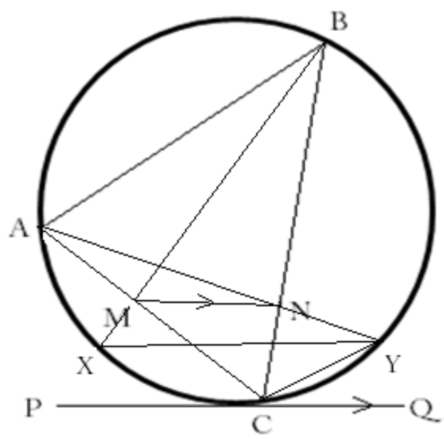
ii) $$\int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\,dx = \int \frac{2(2\sin x + \cos x) + 3(2\cos x - \sin x)}{2\sin x + \cos x}\,dx$$

$$= 2\int dx + 3\int \frac{2\cos x - \sin x}{2\sin x + \cos x}\,dx$$ **1**

$$= 2x + 3\ln(2\sin x + \cos x) + C$$ **1**

(-1 for missing reason)



$\angle ABC = \angle ACP$ (*angles in the alternate segment*)
$\angle NMC = \angle ACP$ (*alternate angles*, $MN \,//\, PC$)
$\therefore \quad \angle ABC = \angle NMC$ **1**

$\angle AMN + \angle NMC = 180°$ (supplementary angles)
$\therefore \quad \angle ABC + \angle AMN = 180°$

$\therefore$ ABNM is a cyclic quadrilateral (sum of interior opposite angles = 180°) **1**

Hence $\angle MAN = \angle MBN$ (*angles in the same segment*) **1**

Also $\angle MBN = \angle XYC$ (*angles in the same segment*)
and $\angle MAN = \angle YCQ$ (*angle in the alternate segment*)

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$\therefore \quad \angle XYC = \angle YCQ$ **1**

Hence XY // PCQ (alternate angles are equal in parallel lines).