Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS EXTENSION 1

HSC - ASSESSMENT TASK 3

Thursday 23rd May, 2013

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 60 minutes

INSTRUCTIONS: This paper consists of 8 questions
ANSWER all questions
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

- ✓ HE1 appreciates interrelationships between ideas drawn from different areas of mathematics
- ✓ HE4 uses the relationship between functions and inverse functions.
- ✓ PE3 solves problems involving permutations and combinations.
- ✓ H5 applies appropriate techniques from the study of series and sequences to solve problems
- ✓ H2 constructs arguments to prove and justify results

	MARKS
Multiple Choice	/5
QUESTION 1 Inverse Trigonometric Functions	/15
QUESTION 2 Permutations and Combinations	/10
QUESTION 3 Sequences and series	/6
	/36

This assessment task constitutes 15% of the Higher School Certificate Course Assessment.

Section 1

5 marks
Attempt Questions 1 – 5
Allow about 5 minutes for this section

Select the alternative A, B, C and D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	**B**	**C**	**D**
1				
2				
3				
4				
5				

1 What is the total number of different eight – letter arrangements that can be formed using the letters in the word " BELLEVUE"?

A 64

B 3 360

C 6 720

D 40 320

2 Which of the following is the value of $\int_0^3 \frac{dx}{(1-x)^2}$?

A $-\frac{3}{2}$

B $-\frac{1}{2}$

C $\frac{1}{2}$

D $\frac{3}{2}$

3 What is the coordinates of the point which divides the interval A(4, – 1) and B (– 4, 2) in the ratio 1 : 3.

A $0, \frac{1}{2}$

B $0, \frac{3}{4}$

C $2, -\frac{1}{4}$

D $-2, \frac{1}{4}$

4 Which of the following is the general solution for $2cosx = \sqrt{3}$?

A $2n\pi \pm \frac{\pi}{3}$

B $2n\pi \pm \frac{\pi}{6}$

C $n\pi + (-1)^n \frac{\pi}{3}$

D $n\pi + \frac{\pi}{6}$

5 If $t = tan\frac{\theta}{2}$, which of the following will be achieved when $\frac{1-cos\theta}{sin\theta}$ is simplified by using t-formula ?

A $\frac{1}{tan\frac{\theta}{2}}$

B $tan\frac{\theta}{2}$

C $1 - tan\frac{\theta}{2}$

D $cot\frac{\theta}{2}$

Section II

31 marks
Attempt Questions 6 – 8
Allow about 55 minutes for this section

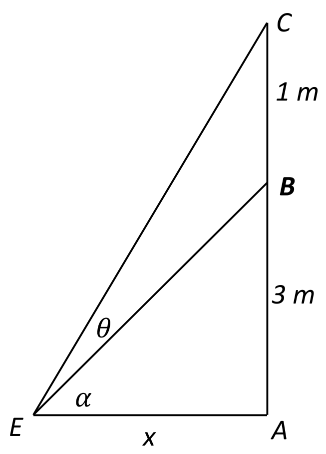
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 6 (15 marks) **Begin a new booklet** **Marks**

1 a) Determine the exact value of $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$ (2)

b) Iris is admiring a tapestry which is hung on a wall with its bottom edge 3 metres above her eye-level E. Let the distance EA be x metres and let θ be the angle that the tapestry subtends at E.



i Show that $\theta = tan^{-1}\frac{4}{x} - tan^{-1}\frac{3}{x}$ (2)

ii Show that θ is maximum when Iris is $2\sqrt{3}$ metres away from the wall. (2)

c) Consider the function $f(x) = 2\cos^{-1}\left(\dfrac{x+1}{3}\right)$.

i) Sketch the curve $f(x)$ showing clearly the coordinates of the end points. (3)

ii Find the equation of the inverse function $f^{-1}(x)$ and hence state its domain and range. (3)

d) Find the exact area bounded by the curve $f(x) = \sin^{-1} x$, x-axis and the lines $x = 1$ and $x = \dfrac{\sqrt{3}}{2}$. (3)

Question 7 (10 marks) **Begin a new booklet** **Marks**

a) Four boys and four girls are arranged in a circle. Find how many ways this can be done :

i) If there are no restrictions (1)

ii) If the boys and girls are in distinct groups (that is 4 boys together and 4 girls together). (1)

iii) If a particular boy and particular girl wish to sit together. (2)

iv) If two particular boys do not wish to sit next to one another. (2)

v) If one particular boy wants to sit between two particular girls. (2)

b) In how many ways can a boat crew of eight women be arranged if three of the women can only row on the bow side and two others can only row on the stroke side? (2)

Question 8 (6 marks) **Begin a new booklet** **Marks**

(a) Anna deposits \$500 at the beginning of each year in a superannuation fund for 9 years. Compound interest is paid at 5%p.a., compounded yearly. Indeed Anna missed the 3rd deposit. Because of that she paid \$1000 at the beginning of the 4th year. How much will the investment be worth at the end of the 9th year? (2)

(b) When Annie finished university with honours, her parents gave her \$20 000. She decided to invest it all in an account which pays 10% p.a compounding yearly. Each year she decided to withdraw \$2500 to go on holiday.

i) Show that after the second holiday, the account will contain (1)

$$\$[(20\,000 \times 1.1^2 - 2500(1 + 1.1)]$$

ii) Find how many years she can continue to withdraw \$2 500 for her holiday. (3)

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Start here Multiple Choice

1. $\frac{8!}{3!\,2!} = 3360$ (B)

2. $\int_0^3 (1-x)^{-2}\,dx = -\int_0^3 (1-x)^{-2}(-dx)$ (A)

$= \left[-\frac{-1}{1-x}\right]_0^3 = \left[\frac{1}{1-3} - 1\right] = -\frac{3}{2}$

3. $\left(2, -\frac{1}{4}\right)$ (C)

4. $\cos x = \frac{\sqrt{3}}{2}$ (B)

$x = 2n\pi \mp \frac{\pi}{6}$

5. $\dfrac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{1-t^2-1+t^2}{2t} = t$ (B)

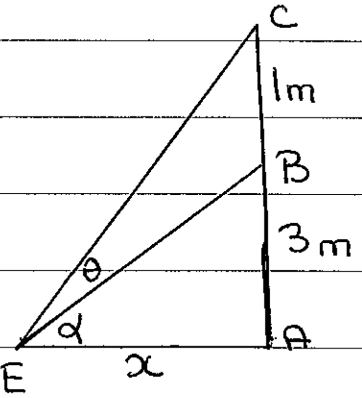
Section 2

6. a) $\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= -\frac{\pi}{3}$

2 marks for correct solution

1 mark substitutes $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$



b i) ΔACE

$\tan(\theta+\alpha) = \frac{4}{x}$

$\theta+\alpha = \tan^{-1}\frac{4}{x}$

ΔABE $\tan\alpha = \frac{3}{x}$

$\alpha = \tan^{-1}\frac{3}{x}$

$\therefore (\theta+\alpha) - (\alpha) = \theta = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{3}{x}$

2 marks correct working out

1 mark shows $\tan\alpha$ or $\tan(\theta+\alpha)$

ii)

Question 6 continues ...

ii) At maximum point $\frac{d\theta}{dx}=0$

Using $\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$

2 marks correct answer

$\frac{d\theta}{dx}=\frac{1}{1+\frac{16}{x^2}}\times\left(\frac{-4}{x^2}\right)-\frac{1}{1+\frac{9}{x^2}}\times\left(\frac{-3}{x^2}\right)$

1 mark differentiates θ with respect to x or any attempt towards solution

$\frac{d\theta}{dx}=\frac{-4}{x^2+16}+\frac{3}{x^2+9}$

$=\frac{-4x^2-36+3x^2+48}{(x^2+16)(x^2+9)}$

$\frac{d\theta}{dx}=\frac{-x^2+12}{(x^2+16)(x^2+9)}$ at stationary pt $\frac{d\theta}{dx}=0$

$x^2=12$, $x=2\sqrt{3}>0$

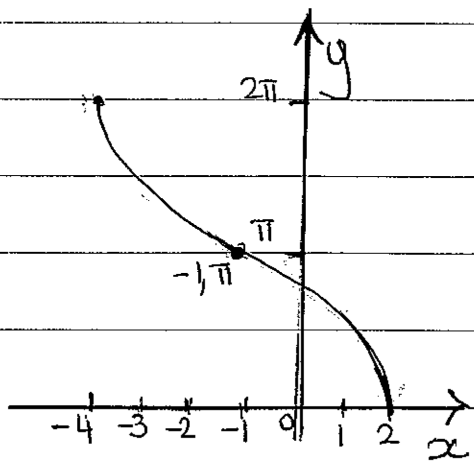
The nature of turning pt:

$\frac{d^2\theta}{dx^2}=\frac{-4(-2x)}{(x^2+16)^2}+\frac{-3(2x)}{(x^2+9)^2}=\frac{8x}{(x^2+16)^2}+\frac{-6x}{(x^2+9)^2}$

When $x=2\sqrt{3}$ $\frac{d^2\theta}{dx^2}=\frac{8\times2\sqrt{3}}{(12+16)^2}-\frac{6\times2\sqrt{3}}{(12+9)^2}$

$\frac{d^2\theta}{dx^2}=-0.01178$ $\therefore$ θ is maximum when $x=2\sqrt{3}$.

c) $f(x)=2\cos^{-1}\left(\frac{x+1}{3}\right)$



1 mark states domain and range

Domain $-1\leqslant\frac{x+1}{3}\leqslant1$

$-3\leqslant x+1\leqslant3$

$-4\leqslant x\leqslant2$

1 mark sketches correct curve, correct end points

Range $0\leqslant\cos^{-1}\left(\frac{x+1}{3}\right)\leqslant\pi$

$0\leqslant2\cos^{-1}\left(\frac{x+1}{3}\right)\leqslant2\pi$

Additional writing space on back page.

Question 6 continues

c ii) $f(x) = 2\cos^{-1}\left(\frac{x+1}{3}\right)$

Inverse function

$x = 2\cos^{-1}\left(\frac{y+1}{3}\right)$

$\frac{x}{2} = \cos^{-1}\left(\frac{y+1}{3}\right)$

$\frac{y+1}{3} = \cos\frac{x}{2}$

$y+1 = 3\cos\frac{x}{2}$

$\therefore \quad f^{-1}(x) = 3\cos\frac{x}{2} - 1$

1 mark find equation of inverse function

1 mark for correct domain and range

Range of $f(x)$ is domain of $f^{-1}(x)$

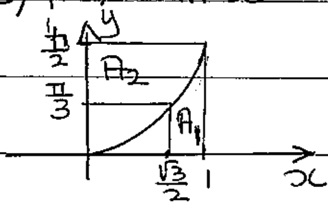
$0 \leq \frac{x}{2} \leq \pi \quad , \quad 0 \leq x \leq 2\pi$

Domain of $f(x)$ is range of $f^{-1}(x)$

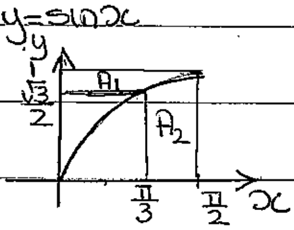
$-1 \leq \cos\frac{x}{2} \leq 1 \quad , \quad -3 \leq 3\cos\frac{x}{2} \leq 3$

$-4 \leq 3\cos\frac{x}{2} \leq 2$

d) $f(x) = \sin^{-1}x$



$y = \sin x$



$A_1 = A_{large\,\square} - A_{small\,\square} - A_2$

$x = 1 \quad f(x) = \sin^{-1}1 = \frac{\pi}{2}$

$x = \frac{\sqrt{3}}{2} \quad f(x) = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$A_{large\,\square} = \frac{\pi}{2} \times 1$ sq. unit

$A_{small\,\square} = \frac{\pi}{3} \times \frac{\sqrt{3}}{2}$

$= \frac{\pi\sqrt{3}}{6}$ sq unit

$A_2 = \int_{\pi/3}^{\pi/2} \sin x \, dx$

$A_2 = \left[-\cos x\right]_{\pi/3}^{\pi/2}$

$A_2 = -\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\right]$

$A_2 = -\left[0 - \frac{1}{2}\right]$

$A_2 = \frac{1}{2}$ square units

1 mark for finding one of the area of rectangles

1 mark finding A_2

1 mark Correct exact answer with working out

You may ask for an extra Writing Booklet if you need more space.

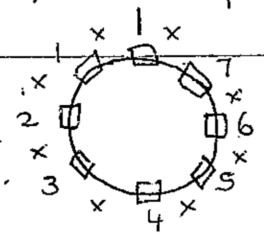
Q6 'd' continues

$$A_2 = \frac{\pi}{2} - \frac{\pi\sqrt{3}}{6} - \frac{1}{2}$$

$$A_2 = \frac{3\pi - \pi\sqrt{3} - 3}{3} \text{ sq units}$$

Q7 a)

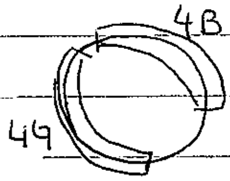
i) Number of arrangements $= 7! \times 1$
$= 5040$



1 mark correct answer with working out.

"Bald correct answer does not attract any mark."

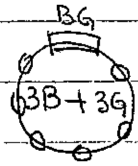
ii)



1 mark correct answer with working out

Number of arrangements $= 4! \times 4! \times 1$
$= 576$

iii)



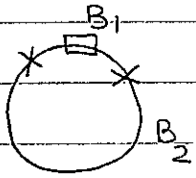
$2! \times 6! \times 1$

2 marks correct solution

No of arrangements $= 2! \times 6! \times 1$
$= 1440$

1 mark shows either of the 2 distinct parts $2!$ or $6!$

iv)



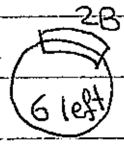
$1 \times 5 \times 6!$
B_1 B_2

2 marks correct solution

Number of arrangements $= 1 \times 5 \times 6!$
$= 3600$

1 mark shows either of the 2 distinct parts.
5 options for B_2
or $6!$ for the rest of the group.

OR If 2 Boys sit together

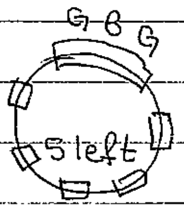


$1 \times 2 \times 6! = 1440$

No of arrangements $= 5040 - 1440$
$= 3600$

(Unrestricted $- 1440$)

Q7 continues

v) 

$2 \times 5! \times 1$

2 marks correct solution
1 mark shows either one of the 2 distinct parts

b) $\overbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}}^{4 \times 3 \times 2}$ Bow

$\underbrace{\underline{\quad}\ \underline{\quad}}_{4 \times 3}$ stroke

remaing seats $3!$

2 mark correct solution

1 mark shows either $^4P_3$ or $^4P_2$ or $3!$

Number of arrangements $= 4 \times 3 \times 2 \times 4 \times 3 \times 3!$
$= 1728$

OR $^4P_3 \times {}^4P_2 \times 3! = 1728$

Q8 a) $r = 5\%$ pa $= 0.05$

A : Amount each deposit grows.
A_1 : Amount 1st deposit grows in 9 years
A_2 : Amount 2nd deposit grows in 8 years
⋮

$\therefore$

$A_1 = 500 \times 1.05^9$ 1st deposit at the end of 9 years
$A_2 = 500 \times 1.05^8$ 2nd deposit at the end of 9 years
$A_3 = 500 \times 1.05^7$
$A_4 = 500 \times 1.05^6$
⋮
$A_9 = 500 \times 1.05^1$

2 marks Correct solution

$A_1 + \cdots + A_9 = 500\underbrace{(1.05 + 1.05^2 + \cdots + 1.05^8 + 1.05^9)}_{\text{Geometric series}}$ $a = 1.05$, $r = 1.05$

$\Sigma A = 500 \dfrac{1.05(1.05^9 - 1)}{1.05 - 1}$

$= \$5788.94$

1 mark correct reasoning for excluding A_3 and A_4 or adding $\$1000 \times 1.05^6$

Q8 continues

But she missed A_3 and invested \$1000 in 4th year. Therefore the amount at the end of 9 years

$A = 5788.94 - 500 \times 1.05^7 - 500 \times 1.05^6 + 1000 \times 1.05^6$

$A = \$5755.44$

b) i) $r = 10\% = 0.1$

Amount at the end of 1^{st} year:

$A_1 = 20000(1+0.1) - 2500$

$A_2 = (20000 \times 1.1 - 2500)1.1 - 2500$

$= 20000 \times 1.1^2 - 2500 \times 1.1 - 2500$

$A_2 = \$[20000 \times 1.1^2 - 2500(1.1+1)]$

1 mark correctly shows A_2

ii) Using the result from part (i)

$A_n = 20000 \times 1.1^n - 2500\underbrace{(1 + 1.1 + \cdots + 1.1^{n-1})}_{\text{Geometric progression}}$

$a=1 \quad r=1.1, \; n \text{ terms}$

1 mark for writing A_n using part "i"

$A_n = 20000 \times 1.1^n - 2500 \dfrac{1(1.1^n - 1)}{1.1 - 1}$

$A_n = 20000 \times 1.1^n - 25000(1.1^n - 1)$

A_n should be greater than \$2500 so that she can go on holiday.

$2500 < 20000\,1.1^n - 25000(1.1^n - 1)$

$1 < 8 \times 1.1^n - 10 \times 1.1^n + 10$

$2 \times 1.1^n < 9$

1 mark for reducing the equation into $1.1^n < 9$

1 mark for correct solution

$1.1^n < 4.5$

If $1.1^n = 4.5$

$n \ln 1.1 = \ln 4.5$

$n = \dfrac{\ln 4.5}{\ln 1.1}$

$n = 15.8$

$n = 15 \quad A_{15} = \$4113.76$

$n = 16 \quad A_{16} = \$2025$

$\therefore n = 16$ years