Killara HIGH SCHOOL

CONSERVA PROGREDERE

STUDENT NUMBER

MATHEMATICS

EXTENSION 1

ASSESSMENT TASK NO 5

Date 28th August 2013

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 60 minutes

INSTRUCTIONS: This paper consists of 4multiple choice questions
ANSWER all 7 questions
Answer questions 1 – 4 on the multiple choice answer sheet provided
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

HE3 – uses a variety of strategies to investigate mathematical models of situations involving binomial theorem.

	MARKS
MULTIPLE CHOICE	/4
QUESTION 1	/11
QUESTION 2	/9
QUESTION 3	/10
	/34

This assessment task constitutes 10% of the Higher School Certificate Course Assessment.

***Section* 1**

4 marks
Attempt Questions 1 – 4
Allow about 5 minutes for this section

Select the alternative A, B, C and D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				

Section 1

Answer question 1 -4 on the multiple choice paper sheet provided

1 Simplify $(1+x)^n\left(1-\frac{1}{x}\right)^n$

A $\left(x-\frac{1}{x}\right)^{2n}$ B $\left(\frac{1-x^2}{x}\right)^n$ C $\left(x-\frac{1}{x}\right)^n$ D $\left(\frac{(x-1)^n}{x}\right)^n$

2 A machine produces mobile phone components. 95% of the components are found satisfactory. For a sample of 12 components, find the probability that exactly 2 are unsatisfactory.

A 0.0015 B 0.0988 C 0.99 D 0.198

3 Simplify ${}^nC_3+{}^nC_4$

A ${}^{2n}C_7$ B ${}^nC_7$ C ${}^{n+1}C_4$ D ${}^{n+1}C_3$

4 If $\sum_{k=4}^{\infty} 2\, r^{k-3} = 6$, find r.

A $\frac{3}{4}$ B 3 C $\frac{6}{7}$ D $\frac{3}{2}$

Question 5 (11 marks) Use a SEPARATE writing booklet

(a) In the binomial expansion of $\left(x^2 + \frac{a}{x}\right)^6$ the term independent of x is 240. Find the value of a. 3

(b) Does the expansion of $\left(3x + \frac{1}{x}\right)^{20}$ contain a term in x^5? Justify your answer. 2

(c) If r is a positive integer and $1 \le r \le 10$, find the largest value of r which satisfies. 3

$$\binom{10}{r} 3^{10-r} \times 2^r > \binom{10}{r-1} 3^{11-r} \times 2^{r-1}$$

(d) Find the exact value of $\left(\sqrt{5}+2\right)^4 + \left(\sqrt{5}-2\right)^4$ and hence find the value of the integer n, such that $n < \left(\sqrt{5}+2\right)^4 < n+1$ 3

Question 6 (9 marks) Use a SEPARATE writing booklet

(a) (i) Write binomial expansion of $\left(1+3x+\dfrac{3x^2}{2}\right)^4$ by considering $3x+\dfrac{3x^2}{2}$ as one term of the binomial expression. 1

(ii) Find the values of the constants k and n so that the expansions of $\left(1+3x+\dfrac{3x^2}{2}\right)^4$ and $(1+kx)^n$ agree as far as the term in x^2. 4

(b) During a shooting practice, the probability of Steve Sharpshoot hitting target is $\dfrac{2}{3}$. Five shots are fired at a target during a target shooting practice. Find the probability of hitting the target :

(i) at least twice 2

(ii) For the expansion of $(a+b)^5$, use $\dfrac{T_{k+1}}{T_k}=\dfrac{n-k+1}{k}.\dfrac{b}{a}$, 2

hence find out the most likely number of successes?

Question 7 (10 marks) Use a SEPARATE writing booklet

(a) i Write down the binomial expansion of $(1+x)^n$ 1

ii Differentiate $x^3(1+x)^n$ and its binomial expansion 2

iii show that $\displaystyle\sum_{r=0}^{n}(r+3)\ {}^nC_r = (n+6)\,2^{n-1}$ 3

b Consider the expansion of $(1+x)^{2n}$,

show that $\displaystyle\sum_{k=0}^{2n}\binom{2n}{k}\frac{2^{k+1}}{k+1} = \frac{3^{2n+1}-1}{2n+1}$ by finding the integrals. 4

SECTION I MULTIPLE CHOICE

1. $(1+x)^n\left(1-\frac{1}{x}\right)^n = \left[(1+x)\left(1-\frac{1}{x}\right)\right]^n$

$= \left[1-\frac{1}{x}+x-1\right]^n$

$= \left(x-\frac{1}{x}\right)^n$ C

2. $n=12$, $p=0.95$ satisfactory

$q=0.05$ unsatisfactory

X = No unsatisfactory

$P(X=2) = {}^{12}C_2 (0.05)^2 (0.95)^{10}$

$= 0.0988$ B

3. ${}^nC_3 + {}^nC_4 = {}^{n+1}C_4$ C

4. $\sum_{k=4}^{\infty} 2r^{k-3} = 6$

$S_\infty = 6$, $2(r+r^2+r^3+\dots+r^\infty)=6$

$S_\infty = 2\frac{r}{1-r}=6$ where $a=r$

$2r=6-6r$

$8r=6$

$r=\frac{3}{4}$

Q5 a) $\left(x^2+\frac{a}{x}\right)^6$, General term $T_{r+1}={}^6C_r\,(x^2)^{6-r}\left(\frac{a}{x}\right)^r$

$\therefore$ The term independant of "x"

$x^{12-2r}\cdot\frac{1}{x^r}=x^0$

1 mark writing term independant of x.

$12-2r-r=0$, $12=3r$, $r=4$

$\therefore T_5={}^6C_4\,(x^2)^2\left(\frac{a}{x}\right)^4$

${}^6C_4\,a^4=240$

$\frac{6!}{4!\times2\times1}a^4=240$

$15a^4=240$

1 mark work towards solution

$a^4=16$

$a=\mp2$

1 mark correct values of "a".

(b) $\left(3x+\frac{1}{x}\right)^{20}$

General term $={}^{20}C_r\,(3x)^{20-r}\left(\frac{1}{x}\right)^r$

$={}^{20}C_r\,3^{20-r}\cdot x^{20-r}\cdot x^{-r}$

1 mark Term in x^5

Term in x^5: $20-r-r=5$

$2r=15$

$r=7.5$

1 mark finding r and comment.

r is a whole number. $r=7.5$ not possible

Hence there is no term in x^5 in the expansion

Q5 c) $\binom{10}{r}3^{10-r}\times 2^r > \binom{10}{r-1}3^{11-r}\times 2^{r-1}$

✓ $\frac{10!}{(10-r)!\,r!}\times\frac{3^{10}}{3^r}\times 2^r > \frac{10!}{(11-r)!\,(r-1)!}\times\frac{3^{11}}{3^r}\times\frac{2^r}{2}$

1 mark correctly expressing the inequality

$\frac{1}{(10-r)!\,r(r-1)!} > \frac{3}{(11-r)(10-r)!\,(r-1)!\,2}$

✓ $\frac{1}{r} > \frac{3}{(11-r)2}$

1 mark progress towards solution.

$22-2r > 3r$

$22 > 5r$, $r < 4\frac{2}{5}$

✓ $\therefore$ Largest value of r is 4.

1 mark expressing "r", and writing the solution

(d) $(\sqrt{5}+2)^4+(\sqrt{5}-2)^4$, if $\sqrt{5}=a$

✓ $(a+2)^4+(a-2)^4 = a^4+{}^4C_1a^3.2+{}^4C_2a^2.2^2+{}^4C_3a.2^3+{}^4C_4.2^4$

$+a^4-{}^4C_1a^3.2+{}^4C_2a^2.2^2-{}^4C_3a.2^3+{}^4C_4.2^4$

$(a+2)^4+(a-2)^4 = 2(a^4+{}^4C_2a^2.4+16)$

$= 2(a^4+24a^2+16)$

$= 2(25+24\times 5+16)$

$(\sqrt{5}+2)^4+(\sqrt{5}-2)^4 = 322$

1 mark correct expansion

since $2<\sqrt{5}<3$

✓ $0<\sqrt{5}-2<1$

$0<(\sqrt{5}-2)^4<1$

1 mark progress towards solution

As $(\sqrt{5}+2)^4+(\sqrt{5}-2)^4 = 322$

✓ $321 < (\sqrt{5}+2)^4 < 322$

1 mark correct values.

6 (a) $\left(1+3x+\frac{3x^2}{2}\right)^4$ and $(1+kx)^n$

Let $y = 3x + \frac{3x^2}{2}$, ~~then~~;

✓ $(1+y)^4 = {}^4C_0 + {}^4C_1 y + {}^4C_2 y^2 + {}^4C_3 y^3 + {}^4C_4 y^4$ — 1mark correct expansion of $(1+kx)^n$ and $\left(1+3x+\frac{3x^2}{2}\right)^4$

For term containing x^2.

$\left(1+3x+\frac{3x^2}{2}\right)^4 = 1 + 4\left(3x+\frac{3x^2}{2}\right) + 6\left(3x+\frac{3x^2}{2}\right)^2 + \dots$ ①

$(1+kx)^n = {}^nC_0 + {}^nC_1 kx + {}^nC_2 k^2x^2 + \dots\dots$ ②

$\therefore$ Equating coefficients of x and x^2 in ① & ②

${}^nC_1 = n$

✓ $12x = nkx$, $\Rightarrow 12 = nk$ and $k = \frac{12}{n}$ ① — 1mark expressing k interms of n.

✓ $6x^2 + 6 \times 9x^2 = \frac{n!}{(n-2)!\,2!} k^2 x^2$

$6 + 54 = \frac{n(n-1)}{2\times 1} \cdot \frac{144}{n^2}$ where $k^2 = \frac{144}{n^2}$ from ① — 1mark progress towards solution

$120n = (n-1)144$

$10n = (n-1)12$

$12 = 2n$, $n = 6$ then $k = \frac{12}{6} = 2$

✓ $\therefore k = 2$ and $n = 6$ — 1mark correct values of k & n

(b) $p = \frac{2}{3}$, $q = \frac{1}{3}$, $n = 5$, $\left(\frac{1}{3} + \frac{2}{3}\right)^5$

X = Number of hits.

6(b)(i) continue

$P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$P(X \geq 2) = 1 - {}^5C_0\left(\frac{1}{3}\right)^5 - {}^5C_1\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)$ 1 mark

$= 1 - \frac{1}{243} - 5 \cdot \frac{1}{81} \times \frac{2}{3}$

$= \frac{232}{243}$ 1 mark

(ii) Greatest term in binomial expansion $(a+b)^n$

$\frac{T_{k+1}}{T_k} = \frac{n-k+1}{k} \times \frac{b}{a}$ where $n=5$, $b=\frac{2}{3}$; $a=\frac{1}{3}$

$\frac{T_{k+1}}{T_k} = \frac{5-k+1}{k} \times \frac{2/3}{1/3}$ ✓ 1 mark correct expression $\frac{T_{k+1}}{T_k}$

$= \frac{6-k}{k} \times \frac{2}{1}$

$T_{k+1} > T_k$, when $12-2k \geq k$

$3k \leq 12$ ✓

$k \leq 4$

$\therefore T_3 < T_4 = T_5$ as

3 hits $P(X=3) = {}^5C_3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}$ 1 mark correct answer

4 hits $P(X=4) = {}^5C_4\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 = \frac{80}{243}$

Most likely number of hitting the target is 3 or 4. Both are equally likely.

7a (i) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$ | 1 mark correct expansion

(ii) $\sum_{r=0}^{n}(r+3)\,{}^nC_r = (n+6)\,2^{n-1}$

Differentiate $x^3(1+x)^n = x^3({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n)$

$LHS = 3x^2(1+x)^n + nx^3(1+x)^{n-1}$ | 1 mark

$RHS = 3x^2({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n) + x^3({}^nC_1 + 2\,{}^nC_2x + 3\,{}^nC_3x^2 + \dots + n\,{}^nC_nx^{n-1})$ | 1 mark

(iii) Let $x=1$

$LHS = 3(2)^n + n(2)^{n-1}$ | 1 mark writing for $x=1$

$= 3.2.2^{n-1} + n\,2^{n-1}$

$= 2^{n-1}(6+n)$ | 1 mark

$RHS = 3({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n) + ({}^nC_1 + 2\,{}^nC_2 + \dots + n\,{}^nC_n)$

$RHS = (3\,{}^nC_0 + 3\,{}^nC_1 + 3\,{}^nC_2 + \dots + 3\,{}^nC_n) + ({}^nC_1 + 2\,{}^nC_2 + \dots + n\,{}^nC_n)$

$= 3\,{}^nC_0 + 4\,{}^nC_1 + 5\,{}^nC_2 + \dots + (3+n)\,{}^nC_n$

$= \sum_{r=0}^{n}(r+3)\,{}^nC_r$ | 1 mark

$\therefore \sum_{r=0}^{n}(r+3)\,{}^nC_r = (n+6)\,2^{n-1}$

b $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + {}^{2n}C_3 x^3 + \cdots + {}^{2n}C_{2n} x^{2n}$

Integrate both sides

$$\int (1+x)^{2n}\,dx = \int \left({}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + {}^{2n}C_3 x^3 + \cdots + {}^{2n}C_{2n} x^{2n}\right) dx$$

✓ $$\frac{(1+x)^{2n+1}}{2n+1} = {}^{2n}C_0 x + {}^{2n}C_1 \frac{x^2}{2} + {}^{2n}C_2 \frac{x^3}{3} + {}^{2n}C_3 \frac{x^4}{4} + \cdots + {}^{2n}C_n \frac{x^{2n+1}}{2n+1} + C$$

✓ Let $x=0$ $\quad \frac{1}{2n+1} = C$

$$\therefore \frac{(1+x)^{2n+1}}{2n+1} = {}^{2n}C_0 x + {}^{2n}C_1 \frac{x^2}{2} + {}^{2n}C_2 \frac{x^3}{3} + {}^{2n}C_3 \frac{x^4}{4} + \cdots + {}^{2n}C_n \frac{x^{2n+1}}{2n+1} + \frac{1}{2n+1}$$

$$\frac{(1+x)^{2n+1}}{2n+1} = \sum_{k=0}^{2n} {}^{2n}C_k \frac{x^{k+1}}{k+1} + \frac{1}{2n+1}$$

Let $x=2$

✓ $$\frac{(3)^{2n+1}}{2n+1} - \frac{1}{2n+1} = \sum_{k=0}^{2n} {}^{2n}C_k \frac{2^{k+1}}{k+1}$$

$$\frac{3^{2n+1}-1}{2n+1} = \sum_{k=0}^{2n} {}^{2n}C_k \frac{2^{k+1}}{k+1}$$

✓ $$\therefore \sum_{k=0}^{2n} \binom{2n}{k} \frac{2^{k+1}}{k+1} = \frac{3^{2n+1}-1}{2n+1}$$

1 mark correct integration.

1 mark correctly substitutes $x=0$ and writes the expansion
1 mark correctly substitutes $x=2$ and writes the equation
1 mark Correct answer.