Killara HIGH SCHOOL

STUDENT NUMBER

Mathematics Extension 1

HSC ASSESSMENT TASK 3

Week 6 2014

Weighting of task 20%

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In questions 6–7, show relevant mathematical reasoning and/or calculations

Total marks – 35

Section I Pages 2 - 3
5 marks
- Attempt Questions 1 – 5
- Allow about 8 minutes for this section

Section II Pages 4 - 7
30 marks
- Attempt Questions 6 – 7
- Allow about 52 minutes for this section

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

		MARKS
Questions 1 – 5	Objective Response	/5
Question 6	Inverse Functions	/18
Question 7	Mathematical Induction	/3
	Permutations and Combinations	/9
	Total	**/35**

Section I

5 marks
Attempt Questions 1-5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1-5

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1 The point C divides the interval joining A (2j, – k) and B (j, 3k) internally in the ratio 3 : 5. **1**

What are the coordinates of C?

(A) $(\frac{11j}{8}, \frac{3k}{2})$

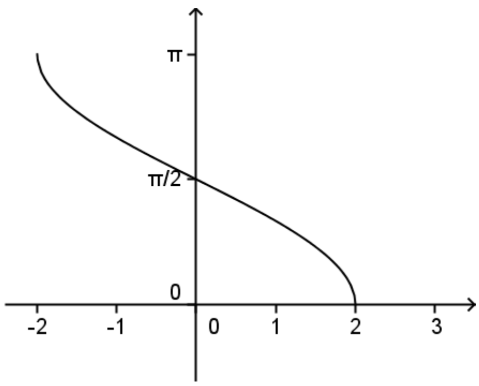
(B) $(\frac{13j}{8}, \frac{k}{2})$

(C) $(\frac{-7j}{8}, \frac{7k}{4})$

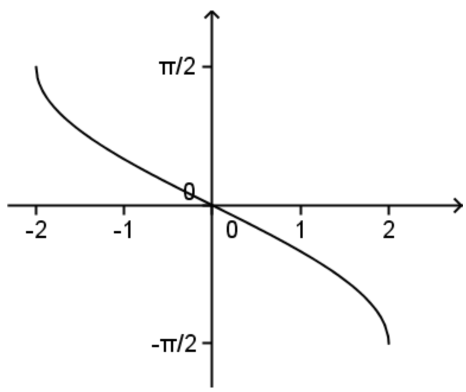
(D) $(\frac{j}{8}, \frac{-9k}{4})$

2 Which diagram shows the graph of $y = \cos^{-1}\frac{x}{2}$? **1**

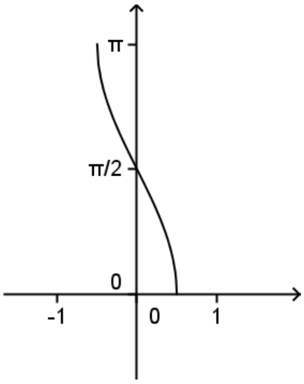
(A)



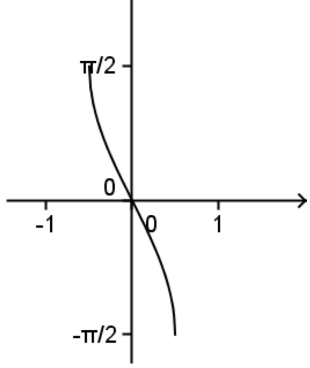
(B)



(C)



(D)



3 Evaluate the $\lim\limits_{x \to 0} \dfrac{\sin 4x}{10x}$ **1**

(A) 0

(B) 0.4

(C) 1

(D) 4

4 For all integers $n \geq 1$, $2^{3n} - 7n - 1$ is divisible by **1**

(A) 0

(B) 49

(C) 98

(D) all of the above

5 Which function does NOT have the same graph as $y = \cos^{-1}(\cos x)$ in the domain $-\pi < x < \pi$? **1**

(A) $y = \sin^{-1}(\sin(x - \frac{\pi}{2})) + \frac{\pi}{2}$

(B) $y = |\tan(\tan^{-1} x)|$

(C) $y = |x|$

(D) $y = \frac{\pi}{2}(1 - \cos x)$

Section II

30 marks
Attempt Questions 6 – 7
Allow about 52 minutes for this section

Answer each question on a SEPARATE sheet of paper.
All necessary working should be shown in every question.

Question 6 (18 marks) Use a SEPARATE sheet of paper. **Marks**

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(a) (i) Sketch the graph of the function $f(x) = \frac{3x^2}{x^2-12}$ showing any intercepts and asymptotes. **3**

(ii) Explain why $f(x)$ does not have an inverse function. **1**

(iii) Show that equation of the inverse relation of f(x) can be written as $y = \pm\sqrt{\frac{12x}{x-3}}$ **1**

(iv) State the domain of the inverse relation $y = \pm\sqrt{\frac{12x}{x-3}}$ **1**

(b) Find $\frac{d}{dx}(x\cos^{-1}2x)$ **2**

(c) Find the volume of the solid formed when the region between $y = (1-25x^2)^{-\frac{1}{4}}$ and the x-axis from $x = -\frac{1}{10}$ to $x = \frac{1}{10}\sqrt{3}$ is rotated about the x-axis. **3**

Question 6 continues

YR12 X1 Task 3 -1- Solutions

Start here Multiple Choice

1. $\dfrac{8!}{3!\,2!} = 3360$ (B)

2. $\displaystyle\int_0^3 (1-x)^{-2}\,dx = -\int_0^3 (1-x)^{-2}(-dx)$ (A)

$= \left[-\dfrac{-1}{1-x}\right]_0^3 = \left[\dfrac{1}{1-3} - 1\right] = -\dfrac{3}{2}$

3. $\left(2, -\dfrac{1}{4}\right)$ (C)

4. $\cos x = \dfrac{\sqrt{3}}{2}$ (B)

$x = 2n\pi \mp \dfrac{\pi}{6}$

5. $\dfrac{1 - \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \dfrac{1-t^2-1+t^2}{2t} = t$ (B)

Section 2

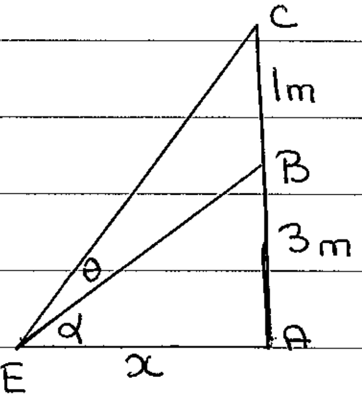
6. a) $\sin^{-1}\left(\sin\dfrac{4\pi}{3}\right) = \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

$= -\dfrac{\pi}{3}$

2 marks for correct solution

1 mark substitutes $\sin\dfrac{4\pi}{3} = -\dfrac{\sqrt{3}}{2}$

b i) ΔACE



$\tan(\theta+\alpha) = \dfrac{4}{x}$

$\theta+\alpha = \tan^{-1}\dfrac{4}{x}$

ΔABE $\tan\alpha = \dfrac{3}{x}$

$\alpha = \tan^{-1}\dfrac{3}{x}$

$\therefore (\theta+\alpha)-(\alpha) = \theta = \tan^{-1}\dfrac{4}{x} - \tan^{-1}\dfrac{3}{x}$

2 marks correct working out

1 mark shows $\tan\alpha$ or $\tan(\theta+\alpha)$

ii)

Question 6 continues...

ii) At maximum point $\frac{d\theta}{dx}=0$

Using $\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$

$\frac{d\theta}{dx}=\frac{1}{1+\frac{16}{x^2}}\times\left(\frac{-4}{x^2}\right)-\frac{1}{1+\frac{9}{x^2}}\times\left(\frac{-3}{x^2}\right)$

$\frac{d\theta}{dx}=\frac{-4}{x^2+16}+\frac{3}{x^2+9}$

$=\frac{-4x^2-36+3x^2+48}{(x^2+16)(x^2+9)}$

$\frac{d\theta}{dx}=\frac{-x^2+12}{(x^2+16)(x^2+9)}$ at stationary pt $\frac{d\theta}{dx}=0$

$x^2=12$, $x=2\sqrt{3}>0$

The nature of turning pt:

$\frac{d^2\theta}{dx^2}=\frac{-4(-2x)}{(x^2+16)^2}+\frac{-3(2x)}{(x^2+9)^2}=\frac{8x}{(x^2+16)^2}+\frac{-6x}{(x^2+9)^2}$

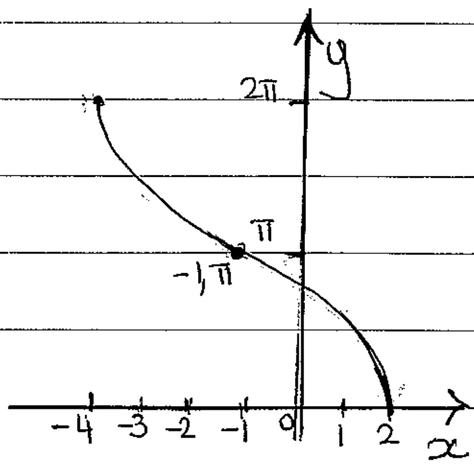
When $x=2\sqrt{3}$ $\frac{d^2\theta}{dx^2}=\frac{8\times2\sqrt{3}}{(12+16)^2}-\frac{6\times2\sqrt{3}}{(12+9)^2}$

$\frac{d^2\theta}{dx^2}=-0.01178$ $\therefore$ θ is maximum when $x=2\sqrt{3}$.

2 marks correct answer

1 mark differentiates θ with respect to x or any attempt towards solution

c) $f(x)=2\cos^{-1}\left(\frac{x+1}{3}\right)$



Domain $-1\leq\frac{x+1}{3}\leq1$

$-3\leq x+1\leq3$

$-4\leq x\leq2$

Range $0\leq\cos^{-1}\left(\frac{x+1}{3}\right)\leq\pi$

$0\leq2\cos^{-1}\left(\frac{x+1}{3}\right)\leq2\pi$

1 mark states domain and range

1 mark sketches correct curve, correct end points

Additional writing space on back page.

Question 6 continues

c ii) $f(x) = 2\cos^{-1}\left(\dfrac{x+1}{3}\right)$

Inverse function

$x = 2\cos^{-1}\left(\dfrac{y+1}{3}\right)$

$\dfrac{x}{2} = \cos^{-1}\left(\dfrac{y+1}{3}\right)$

$\dfrac{y+1}{3} = \cos\dfrac{x}{2}$

$y+1 = 3\cos\dfrac{x}{2}$

$\therefore\ f^{-1}(x) = 3\cos\dfrac{x}{2} - 1$

1 mark find equation of inverse function

1 mark for correct domain and range

Range of $f(x)$ is domain of $f^{-1}(x)$

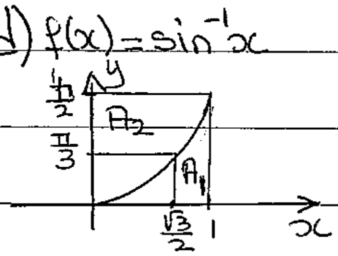
$0 \leqslant \dfrac{x}{2} \leqslant \pi \quad , \quad 0 \leqslant x \leqslant 2\pi$

Domain of $f(x)$ is range of $f^{-1}(x)$

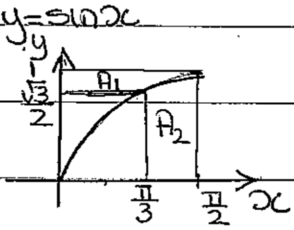
$-1 \leqslant \cos\dfrac{x}{2} \leqslant 1 \quad , \quad -3 \leqslant 3\cos\dfrac{x}{2} \leqslant 3$

$-4 \leqslant 3\cos\dfrac{x}{2} \leqslant 2$

d) $f(x) = \sin^{-1}x$



$y = \sin x$



$A_1 = A_{large} - A_{small} - A_2$

$x = 1 \quad f(x) = \sin^{-1}1 = \dfrac{\pi}{2}$

$x = \dfrac{\sqrt{3}}{2} \quad f(x) = \sin^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}$

$A_{large} = \dfrac{\pi}{2} \times 1$ sq. unit

$A_{small} = \dfrac{\pi}{3} \times \dfrac{\sqrt{3}}{2}$

$= \dfrac{\pi\sqrt{3}}{6}$ sq unit

$A_2 = \displaystyle\int_{\pi/3}^{\pi/2} \sin x\,dx$

$A_2 = \left[-\cos x\right]_{\pi/3}^{\pi/2}$

$A_2 = -\left[\cos\dfrac{\pi}{2} - \cos\dfrac{\pi}{3}\right]$

$A_2 = -\left[0 - \dfrac{1}{2}\right]$

$A_2 = \dfrac{1}{2}$ square units

1 mark for finding one of the area of rectangles

1 mark finding A_2

1 mark Correct exact answer with working out

You may ask for an extra Writing Booklet if you need more space.

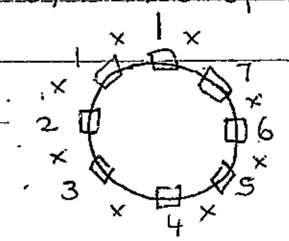
Q6 'd' continues

$A_2 = \frac{\pi}{2} - \frac{\pi\sqrt{3}}{6} - \frac{1}{2}$

$A_2 = \frac{3\pi - \pi\sqrt{3} - 3}{3}$ sq units

Q7 a)

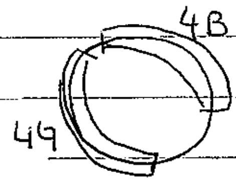
i) Number of arrangements $= 7! \times 1$
$= 5040$



1 mark correct answer with working out.

"Bald correct answer does not attract any mark."

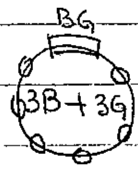
ii)



Number of arrangements $= 4! \times 4! \times 1$
$= 576$

1 mark correct answer with working out

iii)



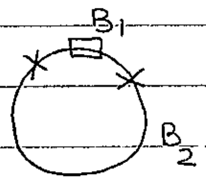
$2! \times 6! \times 1$

No of arrangements $= 2! \times 6! \times 1$
$= 1440$

2 marks correct solution

1 mark shows either of the 2 distinct parts $2!$ or $6!$

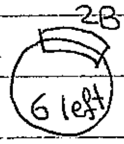
iv)



$1 \times 5 \times 6!$
B_1 B_2

Number of arrangements $= 1 \times 5 \times 6!$
$= 3600$

OR If 2 Boys sit together



$1 \times 2 \times 6! = 1440$

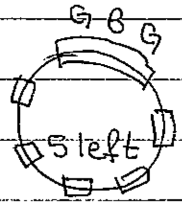
No of arrangements $= 5040 - 1440$
$= 3600$

(Unrestricted − 1440)

2 marks correct solution

1 mark shows either of the 2 distinct parts.
5 options for B_2
or $6!$ for the rest of the group.

Q7 continues

v) 

$2 \times 5! \times 1$

2 marks correct solution
1 mark shows either one of the 2 distinct parts

b) $\overbrace{\boxed{}}^{4 \times 3 \times 2}$ Bow

$\underbrace{\boxed{}}_{4 \times 3}$ stroke

remaing seats $3!$

Number of arrangements $= 4 \times 3 \times 2 \times 4 \times 3 \times 3!$
$= 1728$

OR $^4P_3 \times {}^4P_2 \times 3! = 1728$

2 mark correct solution

1 mark shows either $^4P_3$ or $^4P_2$ or $3!$

Q8 a) $r = 5\%$ pa $= 0.05$

A : Amount each deposit grows.
A_1 : Amount 1^{st} deposit grows in 9 years
A_2 : Amount 2^{nd} deposit grows in 8 years
⋮
∴

$A_1 = 500 \times 1.05^9$ 1^{st} deposit at the end of 9 years
$A_2 = 500 \times 1.05^8$ 2^{nd} deposit at the end of 9 years
$A_3 = 500 \times 1.05^7$
$A_4 = 500 \times 1.05^6$
⋮
$A_9 = 500 \times 1.05^1$

2 marks Correct solution

$A_1 + \cdots + A_9 = 500\underbrace{(1.05 + 1.05^2 + \cdots + 1.05^8 + 1.05^9)}_{\text{Geometric series}}$

$a = 1.05$
$r = 1.05$

$\Sigma A = 500 \dfrac{1.05(1.05^9 - 1)}{1.05 - 1}$

$= \$5788.94$

1 mark correct reasoning for excluding A_3 and A_4 or adding $\$1000 \times 1.05^6$

Q8 continues

But she missed A_3 and invested \$1000 in 4th year. Therefore the amount at the end of 9 years

$A = 5788.94 - 500\times1.05^7 - 500\times1.05^6 + 1000\times1.05^6$

$A = \$5755.44$

b) i) $r = 10\% = 0.1$

Amount at the end of 1^{st} year:

$A_1 = 20000(1+0.1) - 2500$

$A_2 = (20000\times1.1 - 2500)1.1 - 2500$

$= 20000\times1.1^2 - 2500\times1.1 - 2500$

$A_2 = \$[20000\times1.1^2 - 2500(1.1+1)]$

1 mark correctly shows A_2

ii) Using the result from part (i)

$A_n = 20000\times1.1^n - 2500\underbrace{(1+1.1+\cdots+1.1^{n-1})}_{\text{Geometric progression}}$

$a=1 \quad r=1.1, \ n \text{ terms}$

1 mark for writing A_n using part "i"

$A_n = 20000\times1.1^n - 2500\dfrac{1(1.1^n-1)}{1.1-1}$

$A_n = 20000\times1.1^n - 25000(1.1^n-1)$

A_n should be greater than \$2500 so that she can go on holiday.

$2500 < 20000\,1.1^n - 25000(1.1^n-1)$

$1 < 8\times1.1^n - 10\times1.1^n + 10$

$2\times1.1^n < 9$

1 mark for reducing the equation into $1.1^n < 9$

1 mark for correct solution

$1.1^n < 4.5$

If $1.1^n = 4.5$

$n\ln1.1 = \ln4.5$

$n = \dfrac{\ln4.5}{\ln1.1}$

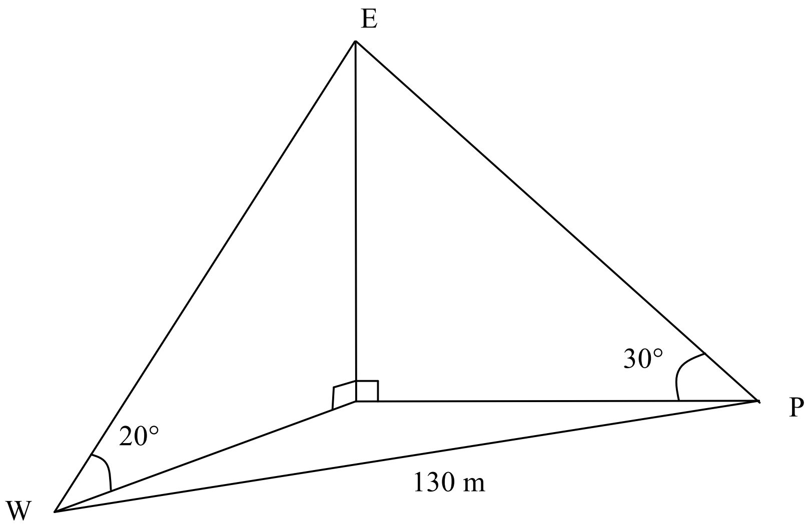
$n = 15.8$

$n=15 \quad A_{15} = \$4113.76$

$n=16 \quad A_{16} = \$2025$

$\therefore n = 16$ years

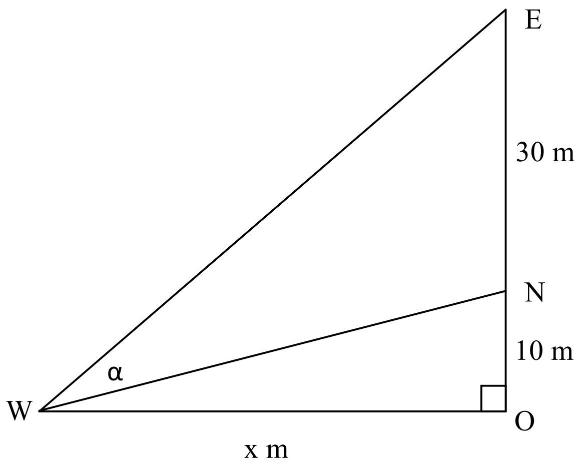
(d) A bird watcher can see an eagle high up in a tall tree due north of him. The angle of elevation to the eagle is 20°. From a different site, 130 metres along a straight, horizontal road, a photographer can see the same eagle, due west of her position, at an angle of elevation of 30°.



i) Show that the eagle is 40 metres above the ground. **3**

Question 6d continues

The bird watcher moves to a new position and can now see an eagle's nest on a branch 10 metres above the level of his eyes and directly below the eagle.



When the bird watcher is standing x metres from the base of the tree, the size of angle ∠EWN is given by

$$\alpha = \tan^{-1}\frac{40}{x} - \tan^{-1}\frac{10}{x}$$

(Do NOT prove this.)

ii) Show that **2**

$$\frac{d\alpha}{dx} = \frac{-40}{x^2 + 1600} + \frac{10}{x^2 + 100}$$

(iii) Hence, find how far from the base of the tree the bird watcher must stand in order for the angle α to be greatest. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Use mathematical induction to show that $5^n > 3^n + 4^n$ for all positive integers $n \geq 3$. **3**

(b) Find the number of ways that 12 different beads can be arranged on a bracelet. **1**

(c) (i) How many eight-letter arrangements of the letters in the word GIGGLING are possible? **1**

(ii) What is the probability that the letters L and N will be next to each other? **2**

(d) In how many ways can 4 boys and 3 girls be arranged in a row if the selection is made from 6 boys and 5 girls? **2**

(e) A supermarket has 5 self-serve checkouts. Two of the checkouts only accept cash while the other three accept cash or credit card. Three shoppers use the checkouts to make their purchases.

(i) One shopper must use a credit card while the other two can use cash or a credit card. Find the number of ways that the three shoppers can make their purchases so that they all use different checkouts. **1**

(ii) On another day, all three shoppers have cash. Find the number of ways in which the three shoppers can make their purchases if at least one uses a cash only checkout and each checkout can be used more than once. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

STUDENT NUMBER

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Select the alternative A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				