Killara HIGH SCHOOL

Student Number

2014

HSC MID COURSE EXAMINATION

Extension 1 Mathematics

11th March 2014

General Instructions

- Reading time – 5 minutes
- Working time 2 hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-14 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

Total Marks – 70

Section I - Pages 2 - 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II - Pages 7 - 10

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Question	Marks
1 - 10	/10
11	/15
12	/15
13	/15
14	/15
Total	/70

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 25% of the Higher School Certificate Course Assessment.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1 – 10

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1 Consider the polynomial $P(x) = 2x^5 - 3x - 7$. Which statement is most correct:

(A) *P(x)* is monic, of degree 5 with a constant term -7

(B) *P(x)* has a leading coefficient 2 , is of degree 5 with a constant term -7

(C) *P(x)* has a leading coefficient 2 , is of degree 5 with a constant term 7

(D) *P(x)* has a leading coefficient 2 , is of degree 3 with a constant term -7

2 If $(x-2)$ is a factor of the polynomial $P(x) = x^3 + 2x^2 - 5x + k$ then

(A) $k = 10$

(B) $k = -10$

(C) $k = 6$

(D) $k = -6$

3 Which integral is obtained when the substitution $u = 2 - x$ is applied to
$\int x\sqrt{2-x}\,dx$

(A) $\int (2-u)\sqrt{2-u}\,du$

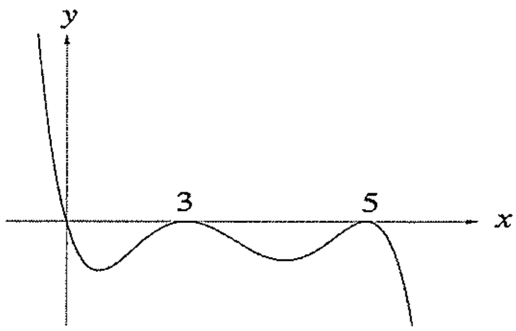
(B) $\int (2-u)\sqrt{u}\,du$

(C) $\int (u-2)\sqrt{u}\,du$

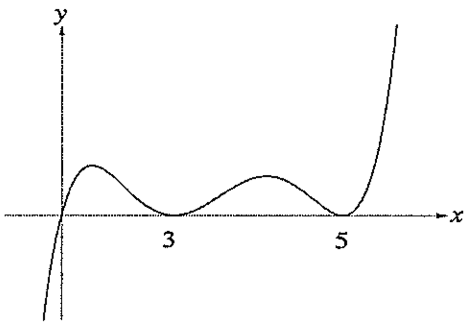
(D) $-\int (u-2)\sqrt{u}\,du$

4 Which diagram best represents the graph $y = x(3-x)^3(x-5)^2$

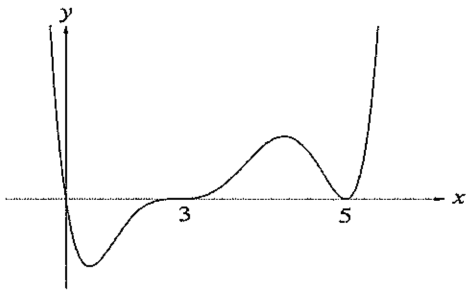
(A)



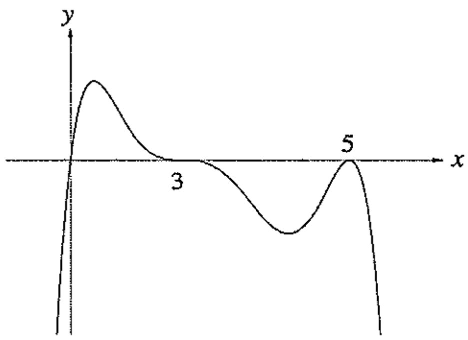
(B)



(C)



(D)



5 Let $|a| \leq 1$. What is the general solution of $\cos 3x = a$?

(A) $x = 2n\pi \pm \dfrac{\cos^{-1} a}{3}$

(B) $x = \dfrac{2n\pi \pm \cos^{-1} a}{3}$

(C) $x = n\pi \pm \dfrac{\cos^{-1} a}{3}$

(D) $x = \dfrac{n\pi \pm \cos^{-1} a}{3}$

6 Which of the following best describes the curve whose parametric equations are:

$$x = \frac{1}{2}\cos\theta, \quad y = \frac{1}{2}\sin\theta, \quad 0 \leq \theta \leq \pi$$

(A) A circle of radius $\dfrac{1}{2}$

(B) A circle of radius 4

(C) A semi-circle of radius $\dfrac{1}{2}$

(D) A semi- circle of radius 4

7 The coordinates of the point P that divides the interval joining (-1, -6) and (4, -1) internally in the ratio 3 : 2 are

(A) $P\left(-\dfrac{11}{5}, -\dfrac{16}{5}\right)$

(B) P (1, -4)

(C) P (2, -3)

(D) $P\left(-\dfrac{14}{5}, -\dfrac{16}{5}\right)$

8 The exact value of $\cos\left(\dfrac{7\pi}{12}\right)$

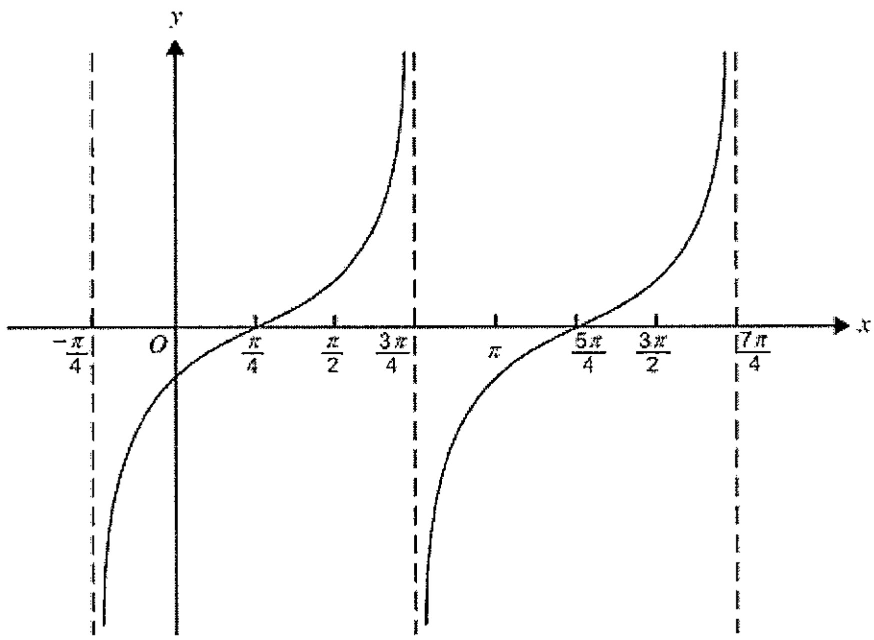
(A) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

(B) $\dfrac{\sqrt{2}+1}{2}$

(C) $\dfrac{\sqrt{2}+\sqrt{6}}{4}$

(D) $\sqrt{6}$

9 The diagram shows two cycles of the graph of a circular function



The period of the circular function is

(A) $\dfrac{\pi}{2}$

(B) $\dfrac{3\pi}{4}$

(C) π

(D) 2π

10 The exact volume when the function $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated around the x-axis is:

(A) $\frac{\pi^2}{4}$

(B) 1

(C) π

(D) $\frac{\pi^2}{2}$

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11(15 marks) Use a SEPARATE writing booklet

(a) The polynomial $P(x) = x^3 + x^2 - kx - 4$ has roots α, β, γ.

(i) Find the value of $\alpha + \beta + \gamma$ **1**

(ii) Find the value of $\alpha\beta\gamma$ **1**

(iii) Find the an expression for $\alpha^2 + \beta^2 + \gamma^2$ **2**

(iv) It is known that two of the roots are equal in magnitude but opposite in sign **2**

Find the third root and hence find the value of k.

(b) Integrate $\int 2\cos^2 x\, dx$ **2**

(c) Prove that **2**

$$\frac{\cos 2\theta}{\cos\theta} + \frac{\sin 2\theta}{\sin\theta} = \frac{4\cos^2\theta - 1}{\cos\theta}$$

(d) (i) Find the vertical and horizontal asymptotes of the hyperbola $y = \dfrac{2x-1}{x-3}$ **3**

and hence sketch the graph of $y = \dfrac{2x-1}{x-3}$

(ii) Hence, or otherwise, find the values of x for which $\dfrac{2x-1}{x-3} \le 4$ **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Integrate $\int \frac{x^2 dx}{\sqrt{x^3+3}}$ using the substitution $u = x^3 + 3$ **2**

(b) Evaluate $\int_1^{25} \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}} dx$ using the substitution $u = 1 + \sqrt{x}$ leaving your solution as an exact value. **3**

(c) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x dx$ **3**

(d) Divide $P(x) = x^4 - 3x^3 + 5x^2 - 8x - 5$ by $x - 3$

Express the result in the form $P(x) = A(x)Q(x) + R(x)$ **3**

(e) Use the substitution $t = \tan\frac{x}{2}$ to solve the equation $5\sin x - 5\cos x + 1 = 0$ for $0 \le x \le 360°$, correct to two decimal places. **4**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Show that $f(x) = x^3 - 2x^2 - x + 1$ has a root between $x = 0$ and $x = 1$ and use the bisection method (halving the interval) twice to determine an estimate of the root. **3**

(b) Use a first approximation of $x = 1.4$ with one application of Newton's method to solve $3\sin x - 2x = 0$ correct to 3 decimal places. **3**

(c) (i) On the same set of axes sketch $y = \sin^2 x$ and $y = \cos^2 x$ for $0 \le x \le 2\pi$ 2

(ii) Show that $y = \sin^2 x$ and $y = \cos^2 x$ intersect at $x = \dfrac{\pi}{4}$ 2

(iii) Find the exact area bounded by the graphs $y = \cos^2 x$, $y = \sin^2 x$ $x = 0$ and $x = \dfrac{\pi}{4}$. 3

(d) A polynomial *P(x)* of degree 3 has zeros at -1, 2 and 3. Its leading coefficient is 4. Write down the polynomial. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) For the curve $y = 5xe^x$

(i) Determine the stationary point and the point of inflexion **3**

(ii) Sketch the curve **2**

(iii) From your sketch, show that the equation $5xe^x = k$ has:

(α) Two roots if $-\frac{5}{e} < k < 0$ **1**

(β) One root if $k > 0$ **1**

(b) (i) Express $\sin 4t + \sqrt{3}\cos 4t$ in the form $R\sin(4t + \alpha)$ where α is in radians **3**

(ii) Hence solve $\sin 4t + \sqrt{3}\cos 4t = 0$ for $0 \le t \le \pi$ **2**

(c) Let $P(x) = (x+2)(x-4)Q(x) + a(x+2) + b$, where *Q(x)* is a polynomial and a and b are real numbers. **3**

When *P(x)* is divided by $(x+2)$ the remainder is -5

When *P(x)* is divided by $(x-4)$ the remainder is 7

What is the remainder when the polynomial is divided by $(x+2)(x-4)$?

End of Paper

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Multiple Choice Answer Sheet (Please detach)

Completely fill the response oval representing the most correct answer.

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

KHS. 2014 Ext1 Half-Yearly.

1. $\boxed{B}$

2. $P(2) = 0$. $\quad 6 + k = 0 \quad k = -6$. $\quad \boxed{D}$

3. $\int x\sqrt{2-x}\,dx$.

$u = 2 - x \Rightarrow x = 2 - u$

$du = -dx$.

$\therefore -\int (2-u)\sqrt{u}\,du = \int (u-2)\sqrt{u}\,du$. $\quad \boxed{C}$.

4. D

5. B.

6. $x = \frac{1}{2}\cos\theta$. $\quad \cos\theta = 2x$ $\quad 0 \le \theta \le \pi$

$y = \frac{1}{2}\sin\theta$. $\quad \sin\theta = 2y$

$\therefore x^2 + y^2 = \frac{1}{4}$.

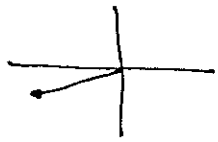
$\boxed{C}$.

7. $(-1,-6) \qquad (4,-1)$.

$3:2$

$$P\left[\frac{3\times 4 + 2\times -1}{3+2}, \frac{3\times -1 + 2\times -6}{3+2}\right] = (2,-3) \quad \boxed{C}.$$

8. $\cos\frac{7\pi}{12}$

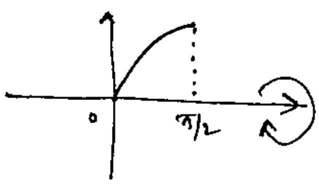


negative answer. $\quad \boxed{A}$.

$\frac{\sqrt{2}-\sqrt{6}}{4}$

9. C

10.



$$\pi\int_0^{\pi/2} \sin^2 x\,dx = \frac{\pi}{2}\int_0^{\pi/2} 1 - \cos 2x\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}\left[\frac{\pi}{2} - 0 - 0 + 0\right] = \frac{\pi^2}{4} \quad \boxed{A}.$$

Question 11

a) $P(x) = x^3 + x^2 - kx - 4$.

(i) $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{1}{1} = -1$

(ii) $\alpha\beta\gamma = -\frac{d}{a} = 4$.

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 1 - 2(-k)$

$= 1 + 2k$.

(iv) $\alpha, -\alpha, \beta$.

$\alpha + -\alpha + \beta = -1 \qquad \therefore \beta = -1$.

$P(-1) = 0. \qquad (-1)^3 + (-1)^2 - k(-1) - 4 = 0$.

$-4 + k = 0$

$\therefore k = 4$.

b). $\int 2\cos^2 x \, dx$

$= 2 \int \frac{1 + \cos 2x}{2} dx$.

$= x + \frac{\sin 2x}{2} + C$.

c) $\frac{\cos 2\theta}{\cos\theta} + \frac{\sin 2\theta}{\sin\theta}$

$\frac{(2\cos^2\theta - 1)\cancel{\sin\theta} + 2\cancel{\sin\theta}\cos\theta \cdot \cos\theta}{\cancel{\sin\theta} \cdot \cos\theta}$

$= \frac{2\cos^2\theta - 1 + 2\cos^2\theta}{\cos\theta}$

$= \frac{4\cos^2\theta - 1}{\cos\theta}$.

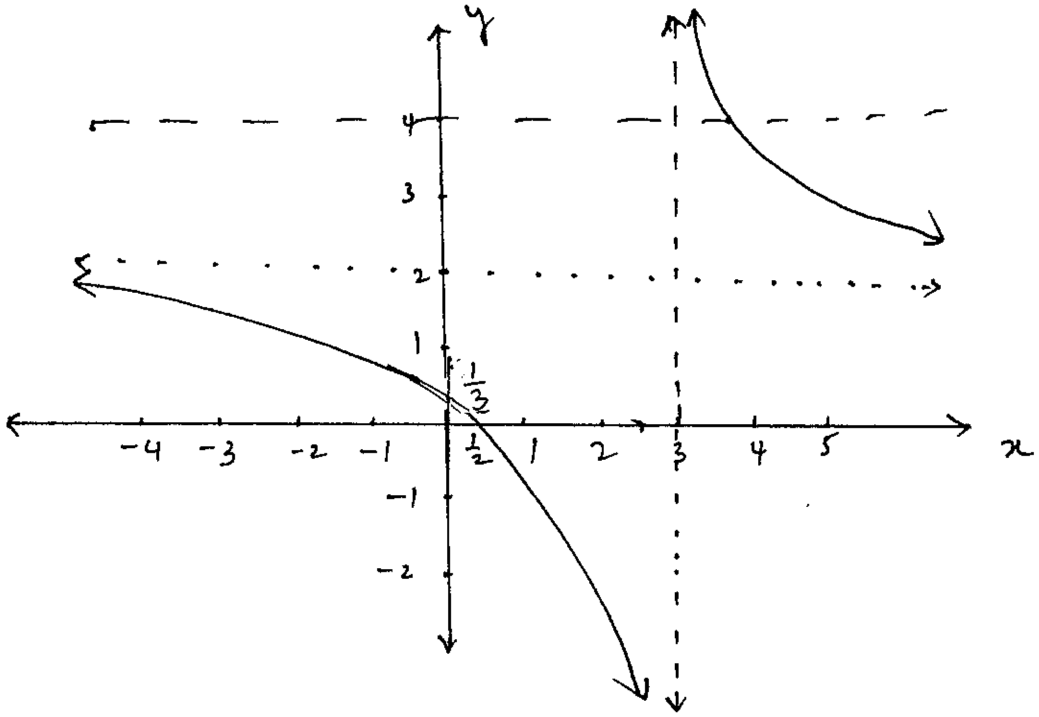
d)

(i) $y = \dfrac{2x-1}{x-3} = \dfrac{2(x-3)+5}{x-3}$

$= 2 + \dfrac{5}{x-3}$

Vertical asymptote $x = 3$.

Horizontal asymptote $y = 2$.



(ii) $\dfrac{2x-1}{x-3} = 4$.

$2x - 1 = 4x - 12$

$2x = 11$

$x = 5.5$

$x \geq 5.5$, $x < 3$

Question 12

a) $\int \frac{x^2\,dx}{\sqrt{x^3+3}}$

$x^3+3 = u$

$3x^2\,dx = du$.

$\frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{x^3+3}}$

$= \frac{1}{3}\int \frac{du}{\sqrt{u}}$

$\frac{1}{3}\int u^{-1/2}\,du$

$\frac{1}{3}\times 2\,u^{1/2} + c \quad = \quad \underline{\underline{\frac{2}{3}\sqrt{x^3+3}}} + c$

b) $\int_1^{25} \frac{1}{\sqrt{x}\left(\sqrt{1+\sqrt{x}}\right)}\,dx$.

$u = 1+\sqrt{x}$.

$du = \frac{1}{2\sqrt{x}}\,dx$

$x=1 \quad u=2$

$x=25 \quad u=6$.

$\int_2^6 \frac{2\,du}{u}$

$= 2\left[\ln u\right]_2^6 = 2\ln\frac{6}{2} = \underline{\underline{2\ln 3}}$

c) $\int_0^{\pi/2} \sin^3 x\cos x\,dx$

$\left[\frac{\sin^4 x}{4}\right]_0^{\pi/2} = \frac{1}{4}\left[1-0\right] = \underline{\underline{\frac{1}{4}}}$.

d)
$$
\begin{array}{r|l}
 & x^3 \quad\ \ +5x \ \ +7 \\
x-3 & x^4 - 3x^3 + 5x^2 - 8x - 5 \\
 & x^4 - 3x^3 \\
\hline
 & 0 + 5x^2 - 8x \\
 & 5x^2 - 15x \\
\hline
 & +7x - 5 \\
 & 7x - 21 \\
\hline
 & 16
\end{array}
$$

$\therefore \quad x^4 - 3x^3 + 5x^2 - 8x - 5 \equiv$

$$(x^2+5x+7)(x-3) + 16.$$

e) $t = \tan\frac{x}{2}$

$5\sin x - 5\cos x + 1 = 0.$

$5 \times \frac{2t}{1+t^2} - 5\,\frac{1-t^2}{1+t^2} + 1 = 0.$

$10t - 5 + 5t^2 + 1 + t^2 = 0.$

$6t^2 + 10t - 4 = 0.$

$3t^2 + 5t - 2 = 0.$

$(t+2)(3t-1) = 0.$

$0 \le x \le 360°$

$t = -2 \qquad t = 1/3. \qquad 0 \le \frac{x}{2} \le 180°$

$\frac{x}{2} = 116.57° \qquad \frac{x}{2} = \tan^{-1}\left(\frac{1}{3}\right) = 18.43\ldots$

$x = 36°52'$

$\therefore x = 233.13.$

<u>Test</u> $x = 180°$

$5\sin 180 - 5\cos 180 + 1 = 6 \neq 0.$

<u>Question 13</u>

a) $f(x) = x^3 - 2x^2 - x + 1$

$f(0) = 1 \qquad f(1) = -1$

$f(x)$ is continuous and change of sign between $f(0)$ and $f(1)$. $\therefore$ Solution exists between 0 and 1.

Estimate 1.

$$x_1 = \frac{0+1}{2} = \frac{1}{2}$$

$f\left(\frac{1}{2}\right) = 0.125$; $f(1) = -1.$

Estimate 2

$$x_2 = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$$

b) $x_0 = 1.4$.

$f(x) = 3\sin x - 2x$

$f'(x) = 3\cos x - 2$

$$x_1 = x_0 - \frac{f(1.4)}{f'(1.4)} = 1.5049\ldots$$

$$= \underline{\underline{1.505}}.$$

c)

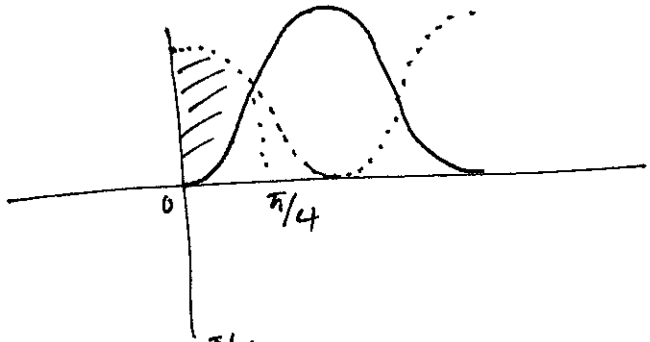
(ii) $x = \pi/4$

$y = \sin^2 \frac{\pi}{4} = \frac{1}{2}$

$y = \cos^2 \frac{\pi}{4} = \frac{1}{2}$

$x = \frac{\pi}{4}$ satisfies both equations.

(iii)



$$\text{Area} = \int_0^{\pi/4} \cos^2 x - \sin^2 x \, dx$$

$$= \int_0^{\pi/4} \cos 2x \, dx$$

$$= \left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[1 - 0\right] = \frac{1}{2} \text{ unit}^2$$

d) $P(x) = 4(x+1)(x-2)(x-3)$.

Question 14

a) $y = 5xe^x$

(i) For stationary point $\frac{dy}{dx} = 0$.

$5[xe^x + e^x] = 0$.

$5e^x[x+1] = 0$ $\quad$ $e^x > 0$ for all x.

$\therefore x = -1$

$x = -1$, $y = -5e^{-1} = -\frac{5}{e} = -1.83$ $\quad (-1, -\frac{5}{e})$.

Point of inflection $\frac{d^2y}{dx^2} = 0$.

$5e^x.1 + 5e^x(x+1) = 0$.

$5e^x[1 + x + 1] = 0$ $\quad$ $e^x > 0$ for all x

$\therefore x = -2$

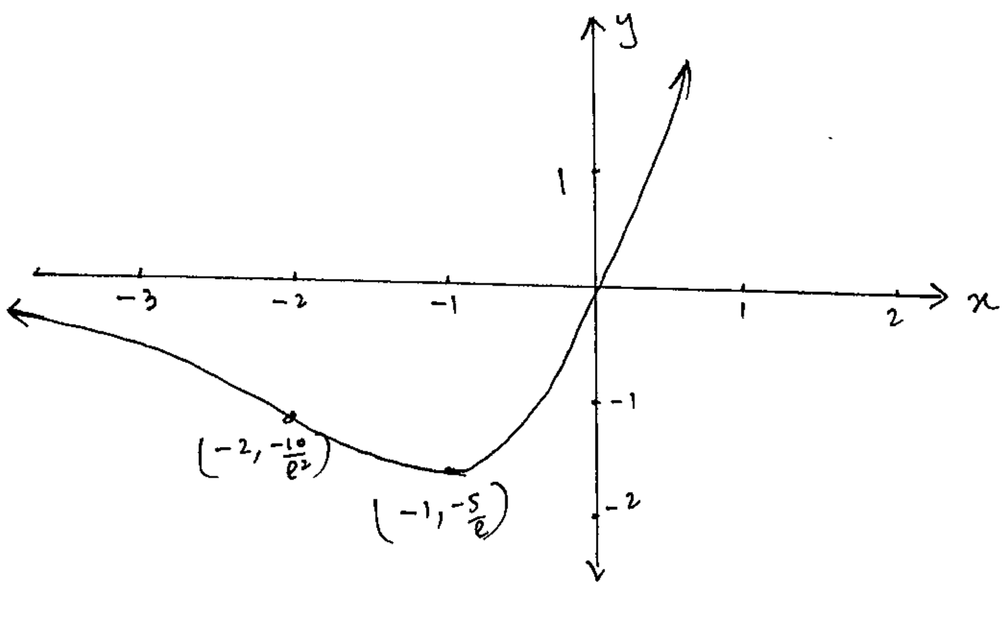
$x = -2$, $y = -10e^{-2} = \frac{-10}{e^2} \doteqdot -1.35$

x	-3	-2	-1
$\frac{d^2y}{dx^2}$	-0.25	0	1.83.

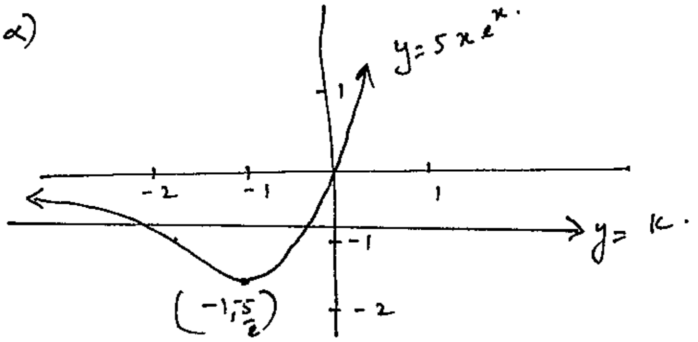
Change in Concavity

$\therefore$ Point of inflection at $(-2, -\frac{10}{e^2})$

(ii) $x \to -\infty$, $y \to 0$; $x \to \infty$, $y = \infty$



(iii) α)

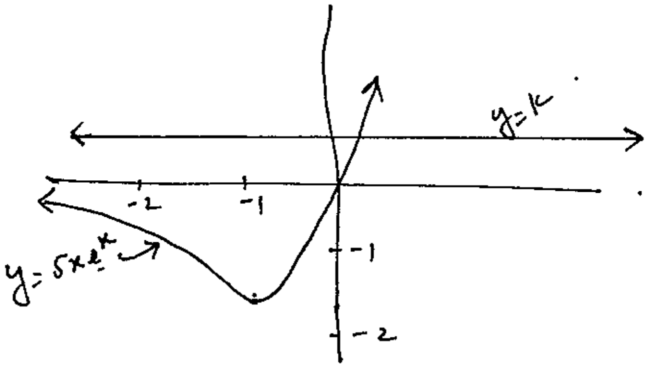


9

For $-\frac{5}{e} < k < 0$, the line $y=k$ cuts the curve $y=5xe^{x}$ at two points.

Hence $5xe^{x}=k$ has two solutions in this region.

β)

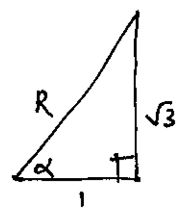


for $k>0$, the line $y=k$ cuts $y=5xe^{x}$ only in one point.

∴ only one solution.

b) (i) $\sin 4t + \sqrt{3}\cos 4t \equiv R\sin(4t+\alpha)$

$R[\sin 4t \cos\alpha + \cos 4t \sin\alpha]$



$\cos\alpha = \frac{1}{R}$ $\quad \sin\alpha = \frac{\sqrt{3}}{R}$

$R=\sqrt{2}$ $\quad \alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

∴ $2\sin\left(4t+\frac{\pi}{3}\right)$.

$0 \le t \le \pi$

$0 \le 4t \le 4\pi$

(ii) $\sin 4t + \sqrt{3}\cos 4t = 0$

$2\sin\left(4t+\frac{\pi}{3}\right) = 0$

$4t+\frac{\pi}{3} = 0, \pi, 2\pi, 3\pi, 4\pi$

$t = \cancel{\frac{\pi}{12}}\ \frac{\pi}{6}, \frac{5\pi}{12}\ \frac{2\pi}{3}, \frac{11\pi}{12}$

$t = \dfrac{\pi - \frac{\pi}{3}}{4}$

c) $P(x) = (x+2)(x-4)Q(x) + a(x+2) + b$.

$P(-2) = -5 \qquad \therefore\ b = -5$

$P(4) = 7 \qquad \therefore\ 6a - 5 = 7$

$6a = 12$

$\therefore\ a = 2$

$\therefore$ Remainder is $a(x+2) + b$

$= 2(x+2) - 5$

$= 2x - 1$