Killara HIGH SCHOOL

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Student Number

# 2015

HIGHER SCHOOL CERTIFICATE
ASSESSMENT 1

# Extension 1 Mathematics

**General Instructions**
- Reading time – 5 minutes
- Working time 60 minutes
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- A Reference Sheet is provided at the back of this paper
- In Questions 5-7 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

## Total Marks – 40

## Section I - Pages 2 - 3
**4 marks**
- Attempt Questions 1 – 4
- Allow about 6 minutes for this section

## Section II - Pages 4- 7
**36 marks**
- Attempt Questions 5 – 7
- Allow about 54 minutes for this section

| Question | Marks |
|---|---|
| 1 - 5 | /4 |
| 6 | /11 |
| 7 | /12 |
| 8 | /13 |
| Total | /40 |

***THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM***

*This assessment task constitutes 10% of the Higher School Certificate Course Assessment*

## Section I

**4 marks**
**Attempt Questions 1 – 4**
**Allow about 6 minutes for this section**

Use the multiple-choice answer sheet for questions 1 – 4 (Detach from paper)

1. What is the value of $\int_0^1 \sqrt{1-y^2}\, dy$ ?

   A. $2\pi$

   B. $\pi$

   C. $\frac{\pi}{2}$

   D. $\frac{\pi}{4}$

2. The Cartesian equation of the curves whose parametric equations are $x = \sin t$ and $y = cos^2 t + 1$ is

   A. $y = x^2 - 1$

   B. $y = 1 - x^2$

   C. $y = 2 - x^2$

   D. $y = x^2 - 2$

3. Which group of three numbers could be the roots of the polynomial equation $x^3 + ax^2 - 41x + 42 = 0$ ?

   A. 2, 3, 7

   B. 1, −6, 7

   C. −1, −2, 21

   D. −1, −3, −14

4. When the polynomial $P(x) = x^4 - 8x^3 - 7x^2 + 3$ is divided by $x^2 + x$, the remainder is $ax + 3$.

What is the value of ?

A. -14

B. -11

C. -2

D. 5

## Section II

**36 marks**
**Attempt Questions 5 – 7**
**Allow about 54 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 5 – 7, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 5** (11 marks) Use a SEPARATE writing booklet

(a) The cubic equation $x^3 + 5x^2 + cx + d = 0$ has three roots; -3, 7 and $\alpha$.

(i) Use the sum of roots to find $\alpha$. **1**

(ii) Find the values of "c" and "d". **1**

(b) Let $P(x) = x^3 + 3x - 7$

(i) Show that the equation $P(x) = 0$ has a root between 1 and 2. **2**

(ii) Use 2 applications of Newton's method with initial approximation $x_1 = 1$ to approximate this root. Give your answer correct to two decimal places. **2**

(c) The polynomial equation $P(x) = 0$ has a double root at $x = a$.

(i) By writing $P(x) = (x-a)^2\,Q(x)$, where $Q(x)$ is a polynomial, show that $P'(a) = 0$. **2**

(iii) Hence or otherwise, find the values of $a$ and $b$ if $x = 1$ is a double root of $x^4 - ax^3 + bx^2 + 5x - 1 = 0$ **3**

**End of Question 5**

**Question 6** (12 marks) Use a SEPARATE writing booklet

(a) A circle is given by the equation $x^2 + y^2 + 4x - 2y - 20 = 0$ **2**
Find the equation of the locus of the mid points of all the chords of length 8 units.

(b) The parametric equations of a curve are $x = t^2 + \frac{1}{t^2}$ and $y = t + \frac{1}{t}$. **1**
Find the Cartesian equation of the curve.

(c) Find a pair of parametric equations for the parabola $x^2 = 12(y - 3)$ **2**

(d) The parametric equations of a parabola are $x = 4t$ and $y = 2t^2$, where $t$ is a parameter.

(i) $P(4p, 2p^2)$ is a point on the above parabola. Find the equation of the tangent to the parabola at $P$. **2**

(ii) $Q$ is the foot of the perpendicular drawn from the vertex of the parabola to the tangent at P.

($\alpha$) Find the coordinates of $Q$. **2**

($\beta$) Show that the locus of $Q$ is given by $y(y^2 + x^2) + 2x^2 = 0$ **3**

**End of Question 6**

**Question 7** (13 marks) Use a SEPARATE writing booklet

(a) Find: $\int (x^3 - \frac{1}{x^2})^2 \, dx$ **2**

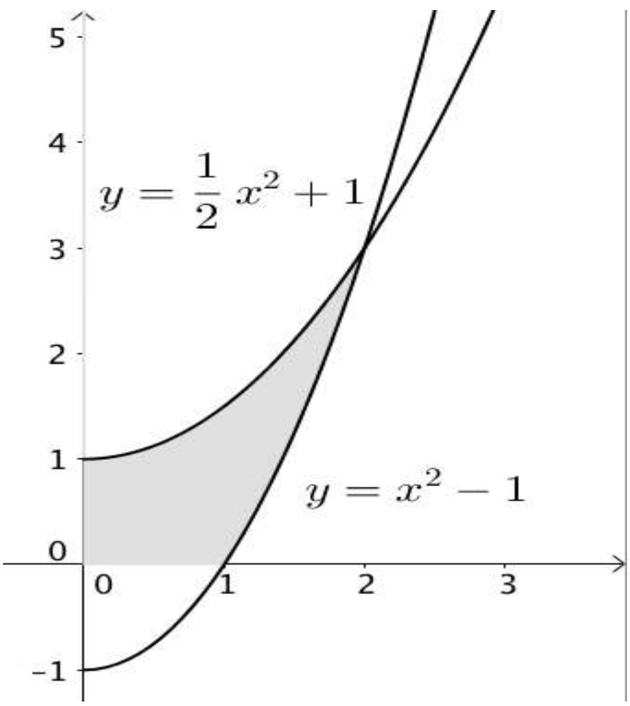
(b) Evaluate: $\int_0^2 \frac{dx}{(3x+2)^2}$ **3**

(c) The following functions $f(x) = 3x^2$ and $g(x) = 7x - x^2$ intersect at the origin and the point A.

(i) Find the coordinates of A where the two curves meet. **2**

(ii) Calculate the area enclosed by the two curves. **3**

(d) A wooden bowl is made by rotating the area between the curves $y = \frac{1}{2}x^2 + 1$ and $y = x^2 - 1$, in the first quadrant about the y-axis.




If the shaded area represents the cross section of the bowl, calculate the volume of wood in the bowl. **3**

# 2015 Task 1 (HSC Ext 1)

(1) Semi circle radius 1

$$\int_0^1 \sqrt{1-y^2}\,dy = \tfrac{1}{2}\pi(1)^2 = \pi/2$$

C

(2) $x = \sin t$ , $y = \cos^2 t + 1$

$= (1-\sin^2 t)+1$

$= 2 - x^2$

C

(3) Could be A or C as needs to be factors of 42 and be positive when multiplied by since there is a term $-41x$ → must have negative

∴

B

(4) By long division

$$\begin{array}{r} x^2 - 9x + 2 \\ x^2+x \,\big)\, x^4 - 8x^3 - 7x^2 \qquad + 3 \\ x^4 + x^3 \qquad\qquad \\ \hline -9x^3 - 7x^3 \qquad \\ -9x^3 - 9x^2 + 0x \\ \hline 2x^2 + 2x \\ \hline -2x + 3 \end{array}$$

C

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$\therefore a = -2$

(5) a) i) $x^3 + 5x^2 + cx + d = 0$

roots $-3, 7, \alpha$

$\therefore -3 + 7 + \alpha = -5$

$\therefore \alpha = -9$ ✓ 1

ii) $c = (-3)(7) + (-3)(-9) + (7)(-9)$ (product of roots two at a time)

$= -57$

$d = -3 \times 7 \times -9 = -189$ ✓ 1

5 b) i) $P(x) = x^3 + 3x - 7$

$P'(x) = 3x^2 + 3$

$P(1) = 1 + 3 - 7 = -3$ ✓ 1

$P(2) = 8 + 6 - 7 = 7$ ✓ 1

Since $P(1) < 0$ and $P(2) > 0$ and $P(x)$ is continuous $1 \leq x \leq 2$ → must be a root between $x = 1$ and $x = 2$

ii) Let $x_1 = 1$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1 - \dfrac{(-3)}{6}$ ✓ 1

$= 1\frac{1}{2}$

$\therefore x_3 = \dfrac{3}{2} - \dfrac{\frac{7}{8}}{\frac{39}{4}}$

$= \dfrac{55}{39}$ ✓ 1

$= 1.41$

c) i) $P(x) = (x-a)^2 Q(x)$ and $P(a) = 0$

$\therefore P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x)$ ✓ 1 by the product rule

$= [x-a]\left(2Q(x) + (x-a)Q'(x)\right)$ ✓ 1

$\therefore P'(a) = 0$ →

ii) $P(x) = x^4 - ax^3 + bx^2 + 5x - 1$ ✓ 1

$P'(x) = 4x^3 - 3ax^2 + 2bx + 5$

Now $P(1) = 1 - a + b + 5 - 1 = -a + b + 5 = 0$ (1)

$P'(1) = 4 - 3a + 2b + 5 = 0$ (2) ✓ 1

from (1) $b = a - 5$ sub into (2)

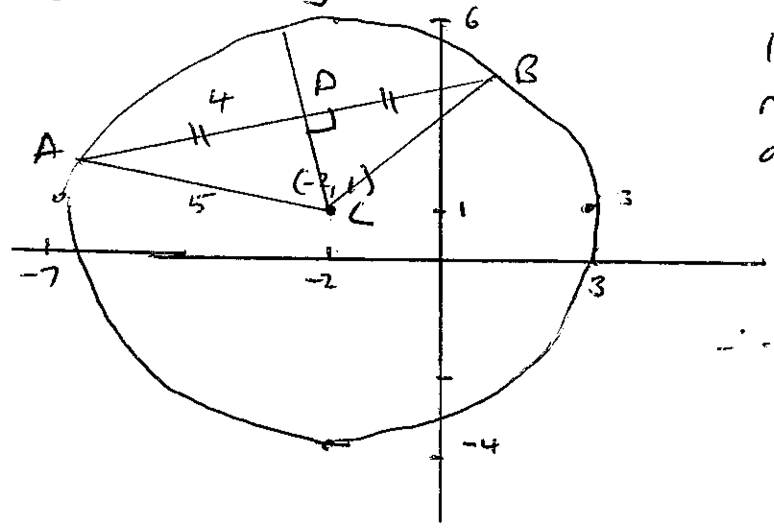
$9 - 3a + 2(a-5) = 0$

$\therefore -a = 1$ ✓ 1

$a = -1$ → $b = -6$

6 a) $x^2 + 4x + 4 \quad + y^2 - 2y + 1 \quad = 20 + 5$ ③

$\therefore (x+2)^2 + (y-1)^2 = 25$




let $AB = 8$
now $AC = 5$
and $CD$ bisects $AB$ at right angles (circle geo!)

$\therefore CD = 3$ by Pythagoras ✓①

$\therefore$ locus of midpoint will be a circle centre $(-2, 1)$ radius 3

✓①

$\therefore (x+2)^2 + (y-1)^2 = 9$

b) $x = t^2 + \frac{1}{t^2}$ $\qquad y = t + \frac{1}{t}$

$\therefore y^2 = \left(t + \frac{1}{t}\right)^2$ ✓①

$= t^2 + \frac{1}{t^2} + \frac{2t}{t}$

$= t^2 + \frac{1}{t^2} + 2$

$\therefore x = y^2 - 2$

c) $x = 6p \;, \quad y = 3(p^2 + 1)$ ✓①

Show $p = \frac{x}{6} \rightarrow y = 3\left(\left(\frac{x}{6}\right)^2 + 1\right)$

$y = \frac{3x^2}{36} + 3$ ✓① for verifying.

$\therefore y - 3 = \frac{x^2}{12}$

$\therefore 12(y-3) = x^2$

6 d) i) $x = 4t$ , $y = 2t^2$ 4

$\rightarrow y = 2\left(\frac{x}{4}\right)^2$

$= \frac{2x^2}{16}$

$= \frac{x^2}{8}$ $\rightarrow$ parabola vertex $(0,0)$

✓ 1

$\therefore \frac{dy}{dx} = \frac{x}{4}$ at $P(4p, 2p^2)$

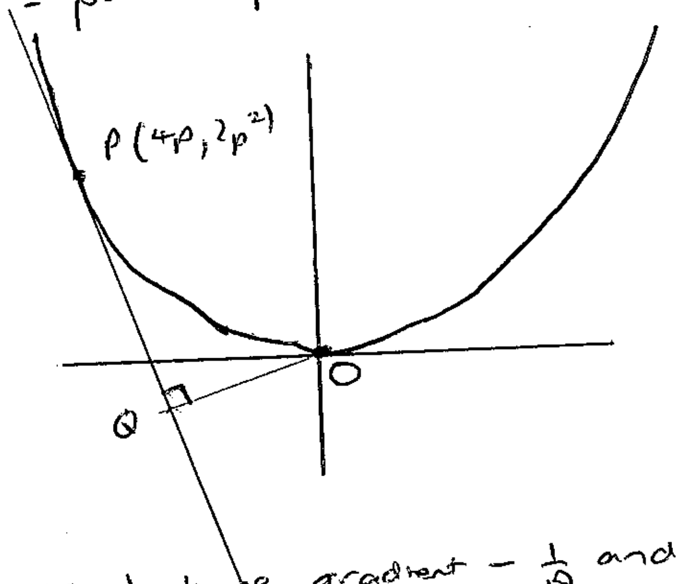
$\frac{dy}{dx} = p$

$\therefore$ Eq$^n$ of tangent

$y - 2p^2 = p(x - 4p)$ ✓ 1

$\therefore y = px - 2p^2$

ii) a)




line OQ must have gradient $-\frac{1}{p}$ and passes through Q

$\therefore y = -\frac{1}{p}x$

find Q by solving simultaneously

$-\frac{1}{p}x = px - 2p^2$

$-x = p^2x - 2p^3$

$\therefore 2p^3 = p^2x + x$

$= x(p^2 + 1)$

$\therefore x = \frac{2p^3}{p^2 + 1}$

$\therefore y = \frac{-2p^2}{p^2 + 1}$

13) $x = \frac{2p^3}{p^2+1}$ $\quad y = \frac{-2p^2}{p^2+1}$ (5)

Notice

$x = -py$

$\therefore p = \frac{-x}{y}$ → substitute ✓ 1

$$x = \frac{2\left(\frac{-x}{y}\right)^3}{\left(\frac{-x}{y}\right)^2+1}$$

$$= \frac{\frac{-2x^3}{y^3}}{\frac{x^2}{y^2}+1}$$ ✓ 1

$$x = \frac{-\frac{2x^3}{y^3}}{\frac{x^2+y^2}{y^2}}$$

$$\therefore x = \frac{-2x^3}{y(x^2+y^2)}$$

$\therefore xy(x^2+y^2) = -2x^3$

$\therefore y(x^2+y^2) = -2x^2$ since $x \neq 0$ ✓ 1

$\therefore y(y^2+x^2) + 2x^2 = 0$

(6)

7 a) $\int \left(x^{3} - \frac{1}{x^{2}}\right)^{2} . dx$

$= \int \left(x^{6} - \frac{2x^{3}}{x^{2}} + x^{-4}\right) . dx$ ✓ 1

$= \int \left(x^{6} - 2x + x^{-4}\right) dx$

$= \frac{x^{7}}{7} - \frac{x^{2}}{\text{[illegible]}} - \frac{x^{-3}}{3} + c$ ✓ 1

b) $\int_{0}^{2} \frac{dx}{(3x+2)^{2}} = \left[-\frac{1}{3}(3x+2)^{-1}\right]_{0}^{2}$

$= \left[-\frac{1}{9x+6}\right]_{0}^{2}$

$= \left[\frac{-1}{24}\right] - \left[-\frac{1}{6}\right]$

$= \frac{1}{8}$

c) $f(x) = 3x^{2}$ , $g(x) = 7x - x^{2}$

$= x(7-x)$

$\therefore$ Intersect when

$3x^{2} = 7x - x^{2}$

$\therefore 4x^{2} - 7x = 0$

$x(4x-7) = 0$

intersect

$x = 0, \quad x = \frac{7}{4}$

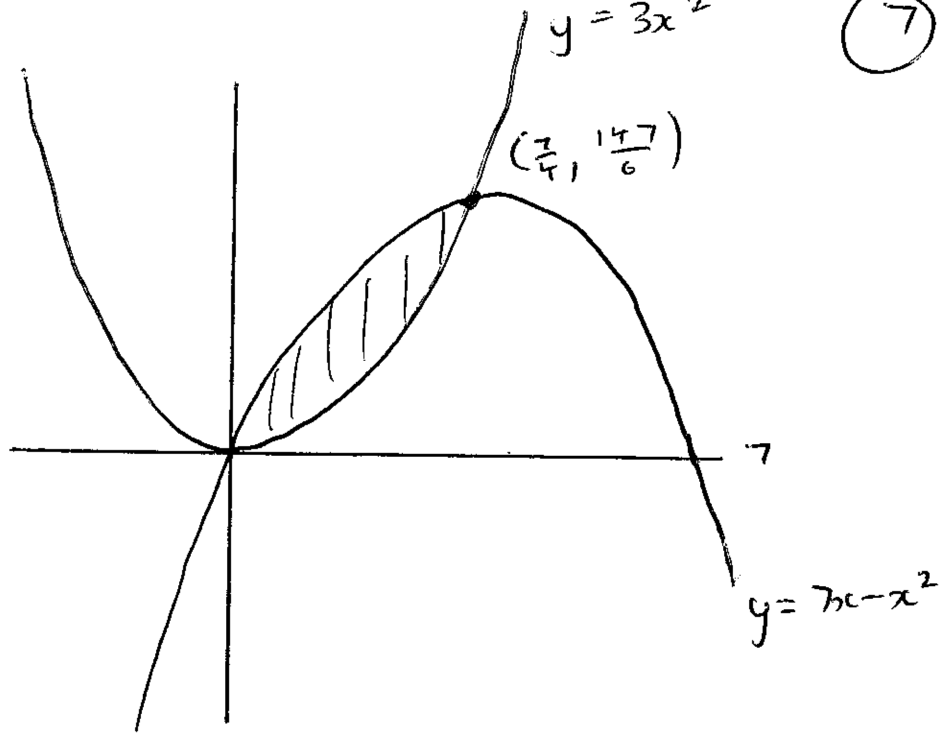
when $x = \frac{7}{4}$ , $y = 3\left(\frac{49}{16}\right) = \frac{147}{16}$

Curves meet at

$\left(\frac{7}{4}, \frac{147}{16}\right)$

7c) continued






$$A = \int_0^{7/4} (7x - x^2) - (3x^2) \, dx$$ ✓ 1

$$= \int_0^{7/4} (7x + 2x^2) \, dx$$

$$= \left[ \frac{7}{2}x^2 + \frac{2}{3}x^3 \right]_0^{7/4}$$ ✓ 1

$$= \left[ \frac{7}{2}\left(\frac{49}{4}\right) + \frac{2}{3}\left(\frac{343}{8}\right) \right] - 0$$

$$= \left[ \frac{343}{8} + \frac{686}{24} \right]$$

$$= \frac{1715}{24} \text{ units}^2$$ ✓ 1

7 d) $y = \frac{1}{2}x^2 + 1$ , $y = x^2 - 1$ (8)

meet when

$x^2 - 1 = \frac{1}{2}x^2 + 1$

$\therefore \frac{1}{2}x^2 = 2$ ✓ (1)

$x^2 = 4$

$x = \pm 2$ → we only want

$x = 2$

$\therefore y = 3$

$$\text{Volume} = \pi \int_0^3 x^2 . dy$$

where $x^2$ is the radius of the disc

Now from the graph the region we need has radius

$y = x^2 - 1$

$\therefore x^2 = y + 1$

$y = \frac{1}{2}x^2 + 1$

$y - 1 = \frac{1}{2}x^2$

$x^2 = 2y - 2$

$\therefore$ radius of disc

$(y+1) - (2y-2)$

OR

$$V = \pi \int_0^3 (y+1) . dy - \pi \int_0^3 (2y-2) dy$$ ✓ (1)

$$= \pi \int_0^3 (-y + 3) . dy$$

$$= \pi \left[ -\frac{y^2}{2} + 3y \right]_0^3$$

$$= \pi \left[ -\frac{9}{2} + 9 \right]$$ ✓ (1)

$$= \frac{9\pi}{2} \text{ units}^3$$