Killara HIGH SCHOOL

Student Number

2015
HSC course
Mathematics
EXTENSION 1
25th May 2015

General Instructions
- Reading time – 5 minutes
- Working time - 1 hour
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Start a new booklet for each question

Total Marks – 45

Question	Marks
1	/14
2	/15
3	/16

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 20% of the Higher School Certificate Course Assessment

	Questions	Marks
1.	a) If $f(x)=2\sin^{-1}3x$ then find:	
	i) the domain and range of $f(x)$.	**2**
	ii) $f\left(\frac{1}{6}\right)$.	**1**
	iii) $f^{1}\left(\frac{1}{6}\right)$.	**2**
	b) For what value of x will the geometric series have an infinite sum?	
	$\left(\frac{x}{2x-1}\right)+\left(\frac{x}{2x-1}\right)^2+\left(\frac{x}{2x-1}\right)^3+\dots\dots\dots\dots$	**2**
	c) How many straight line arrangements can be made from the word BALANCES if:	
	i) there are no restrictions?	**1**
	ii) the letters B and S are at either end?	**2**
	iii) the 3 vowels are next to each other?	**1**
	d) Prove by Mathematical Induction that	**3**
	$1\times2^0+2\times2^1+3\times2^2+\dots\dots+n\times2^{n-1}=1+(n-1)2^n$ for all integers $n\geq1$.	

	Questions	Marks
2		
	a) i) Sketch the function $y = f(x)$ where $f(x) = (x-1)^2 - 4$ clearly showing all intercepts on the coordinate axes (use the same scale on both axes).	2
	ii) What is the largest positive domain of $f(x)$ for which $f(x)$ has an inverse $f^{-1}(x)$?	1
	iii) Sketch the graph of $y = f^{-1}(x)$ on the same number plane as in part i).	1
	b) i) Show that $\cos^{-1}\frac{3}{5} - \tan^{-1}\left(\frac{-3}{4}\right) = \frac{\pi}{2}$	3
	ii) Differentiate $x\tan^{-1}x$.	1
	iii) Hence evaluate $\int_0^1 \tan^{-1}x \; dx$	3
	c) There are 12 DVD's arranged in a row on a shelf in Mrs Jackson's study. There are 5 copies of Superman 1, 4 copies of Avengers and 3 copies of Gladiator.	
	i) How many arrangements of the DVD's are possible?	1
	ii) How many different arrangements are there if the DVD's with the same title are grouped together?	1
	d) Consider the word EXERCISES. What is the probability of an arrangement beginning with the letters CR or RC.	2

	Questions	Marks
3	a) Kyle borrowed \$26 000 (P) at an interest rate of 9% p.a. compounded monthly. She is to repay the loan in 8 equal instalments over 2 years. That is, she will make repayments of $\$R$ at 3 monthly intervals. Let $K=(1.0075)^3$ and A_n = the amount owing on the loan immediately after the nth repayment.	
	i) Show $A_n = PK^n - \dfrac{R(K^n-1)}{K-1}$.	2
	ii) Show that the value of the repayments $\$R$ is \$3590.20.	2
	iii) When Kyle made her 7th repayment she repaid an additional \$2000. That is, she repaid \$5590.20. Determine the amount by which her final repayment will be reduced as a result of the additional \$2000 in her 7th repayment.	2
	b) The region in the first quadrant bounded by the curve $y=2\tan^{-1}x$ and the $y-axis$ between $y=0$ and $y=\dfrac{\pi}{2}$ is rotated about the $y-axis$. Find the exact volume of the solid of revolution formed.	3
	c) Prove by Mathematical Induction that $3^n+4^n<5^n$ for all positive integers $n\geq 3$.	3
	d) Find the number of ways of seating 5 boys and 5 girls around a round table if:	
	i) There is no restriction where each person sits.	1
	ii) A particular girl (L) wishes to sit between 2 particular boys B and G.	1
	iii) Two particular people (G and J) do not wish to sit together.	2
	END OF ASSESSMENT	

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

2015 Solutions Mathematics Ex 1 25th May

Question 1

a) i) Domain: $-1 \le 3x \le 1$
$-\frac{1}{3} \le x \le \frac{1}{3}$

Range: $-\pi \le y \le \pi$

ii) $f(\frac{1}{6}) = 2\sin^{-1}\frac{1}{2}$
$= 2 \times \frac{\pi}{6}$
$= \frac{\pi}{3}$

iii) $f(x) = 2\sin^{-1}3x$
$f'(x) = 2 \times \frac{3}{\sqrt{1-9x^2}}$
$f'(\frac{1}{6}) = \frac{6}{\sqrt{1-9\times\frac{1}{36}}}$
$= 4\sqrt{3}$

i)
1 Domain
1 Range

ii) 1 for correct answer

iii) 1 for correct differentiation
1 for correct substitution

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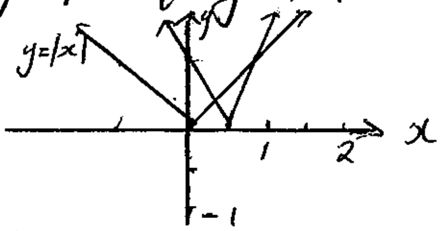
b) $r = \frac{x}{2x-1}$, For sum to infinity
$|r| < 1$
$\left|\frac{x}{2x-1}\right| < 1$
$|x| < |2x-1|$

Consider graph of $y=|x|$ and $y=|2x-1|$



$y=x$ and $y=-2x+1$ intersect at $x=\frac{1}{3}$
$y=x$ and $y=2x-1$ intersect at $x=1$
$\therefore$ $|x| < |2x-1|$ occurs where
$x < \frac{1}{3}$ and $x > 1$

$\therefore$ an infinite sum exists where $x < \frac{1}{3}$ and $x > 1$

1 for showing graphs of $y=|x|$ and $y=|2x-1|$
or
a method that works from $|x| < |2x-1|$
(don't give mark for this line only)

1 for answer

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c)
i No Restrictions $= \frac{8!}{2!}$

ii Arrangements $= \frac{2!\times 6!}{2!} = 6!$

iii When 3 vowel are together $= \frac{6!}{3}$

1 for $\frac{8!}{2!}$

1 for $2!\times 6!$

2 for working to final answer.

1 for answer

Question 1

d) For $n=1$ $LHS=1$, $RHS=1$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $1\times 2^0+2\times 2^1+3\times 2^2+\dots+k\times 2^{k-1}=1+(k-1)2^k$ ①

Required to show for $n=k+1$

$1\times 2^0+2\times 2^1+3\times 2^2+\dots+k\times 2^{k-1}+(k+1)2^k=1+k\times 2^{k+1}$

Now, $LHS = 1\times 2^0+2\times 2^1+\dots+k\times 2^{k-1}+(k+1)2^k$

$= 1+(k-1)2^k+(k+1)2^k$ (sub from ①)

$= 1+2^k[(k-1)+(k+1)]$

$= 1+2^k\times 2k$

$= 1+k\times 2^{k+1}$

$= RHS$

If the result is true for $n=1$ and proven true for $n=k+1$ after assumed to be true for $n=k$ then it is true for all positive integers of n.

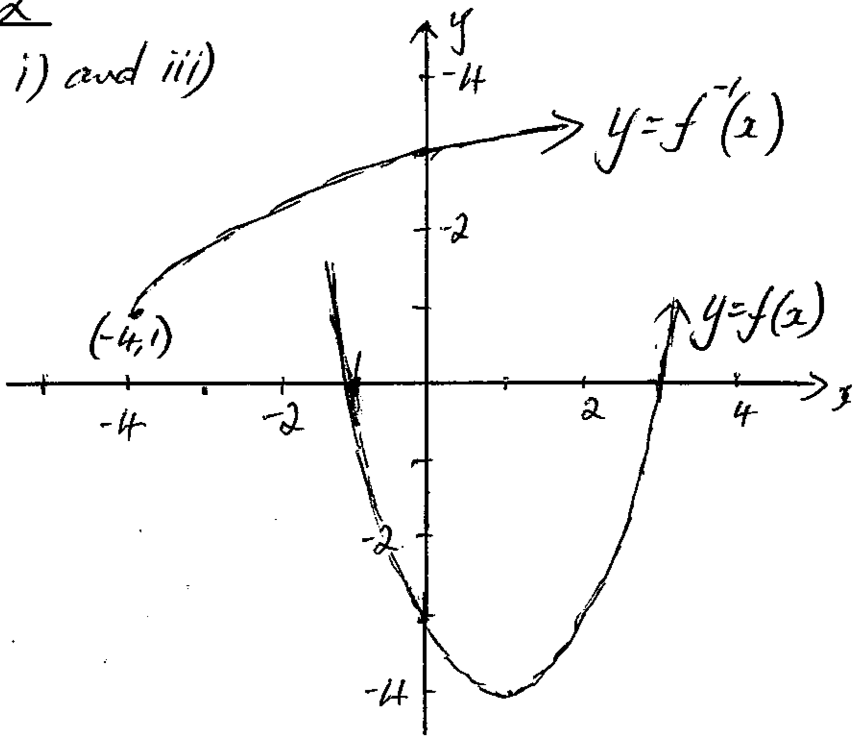
1 for showing true for $n=1$

1 for writing the correct assumption

1 for working through the necessary steps to show true for $n=k+1$

Question 2

a) parts i) and iii)



ii) $x \geqslant 1$

i) 1 mark for shape
1 mark for intercepts. and showing $(1,-4)$

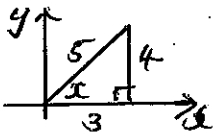
iii) 1 mark for showing $(-4,1)$ and passing through $(0,3)$

ii) 1 for answer

b) i) $\cos^{-1}\frac{3}{5} - \tan^{-1}\left(\frac{-3}{4}\right) = \frac{\pi}{2}$

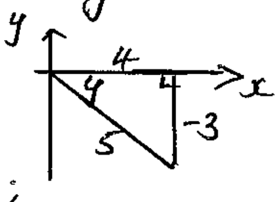
let $x = \cos^{-1}\frac{3}{5}$

$\therefore \cos x = \frac{3}{5}$



let $y = \tan\left(\frac{-3}{4}\right)$

$\therefore \tan y = \frac{-3}{4}$



$\therefore x - y = \frac{\pi}{2}$

Take sine of both sides

ie $\sin(x-y) = \sin\frac{\pi}{2}$

$LHS = \sin(x-y)$

$= \sin x \cos y - \cos x \sin y$

$= \frac{4}{5} \times \frac{4}{5} - \frac{3}{5} \times \left(\frac{-3}{5}\right)$

$= 1$

$RHS = \sin\frac{\pi}{2} = 1$

Thus $LHS = RHS$

$\therefore \cos^{-1}\left(\frac{3}{5}\right) - \tan\left(\frac{-3}{4}\right) = \frac{\pi}{2}$

1 for using x and y to write $\cos x = \frac{3}{5}$ and $\tan y = -\frac{3}{4}$

1 for diagrams and writing $\sin(x-y) = \sin\frac{\pi}{2}$

1 for working through this to show $\cos^{-1}\left(\frac{3}{5}\right) - \tan\left(\frac{-3}{4}\right) = \frac{\pi}{2}$

Question 2

c) i) Number of arrangements $= \frac{12!}{3!4!5!}$ — 1 for answer

ii) Number of arrangements $= 3!$ — 1 for answer

d) Total Number of arrangements $= \frac{9!}{3!2!}$

Number of arrangements starting with CR or RC $= \frac{2!8!}{3!2!}$ — 1 for $\frac{8!}{3!}$

$= \frac{8!}{3!}$

P(start with CR or RC) $= \frac{8!}{3!} \div \frac{9!}{3!2!}$ — 1 for final answer

$= \frac{8! \times 2}{9!}$

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b) ii)

$y = x\tan^{-1}x$

$\frac{dy}{dx} = x\left(\frac{1}{1+x^2}\right) + \tan^{-1}x$

$\frac{d}{dx}(x\tan^{-1}x) = \frac{x}{1+x^2} + \tan^{-1}x$ — 1 for the answer

iii) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \frac{d}{dx}(x\tan^{-1}x)\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$ — 1 for writing the definite integrals

$= \left[x\tan^{-1}x\right]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ — 1 for substitution

$= \tan^{-1}1 - 0 - \left[\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1\right]$ — 1 for answer

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

Question 3

a) i) $A_1 = P(1.0075)^3 - R$

$A_2 = \{P(1.0075)^3 - R\}(1.0075)^3 - R$

$= P(1.0075)^6 - R\left[(1.0075)^3 + 1\right]$

$= PK^2 - R(K+1)$

$A_n = PK^n - R\left(K^{n-1} + K^{n-2} + \cdots + 1\right)$

$= PK^n - R \times \dfrac{a(r^n - 1)}{r-1}$

$= PK^n - R \times \dfrac{1(K^n - 1)}{K-1}$

$= PK^n - \dfrac{R(K^n - 1)}{K-1}$

1 for working towards A_2

1 for writing A_n from A_2 and WORKING through to the final result

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ii) When $n = 8$, $A_n = 0$

$\therefore PK^8 - \dfrac{R(K^8 - 1)}{K-1} = 0$

$\therefore 26{,}000 \times (1.0075)^{24} = \dfrac{R(1.0075^{24} - 1)}{1.0075^{23} - 1}$

$R = \$3590.20$

1 for correct substitution

1 for WORKING towards correct answer

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iii) $A_7 = 26000(1.0075)^{21} - \dfrac{3590.20(1.0075^{21} - 1)}{1.0075^3 - 1}$

$= \$3510.64$

Because Kyle repaid an additional \$2000 for the 7th repayment then the amount owing at the start of the last 3 months is

$3510.64 - 2000 = 1510.64$

Last payment $= 1510.64 \times 1.0075^3$

$= \$1544.88$

Last payment is reduced by $= 3590.20 - 1544.88$

$= \$2045.35$

1 for obtaining \$1510.64

1 for working towards \$2045.32

Question 3

b) $y = 2\tan^{-1}x$

$\frac{1}{2}y = \tan^{-1}x$

$\tan(\frac{1}{2}y) = x$

$$V = \pi\int_0^{\frac{\pi}{2}} \tan^2(\tfrac{1}{2}y)\,dy$$

$$= \pi\int_0^{\frac{\pi}{2}} \sec^2(\tfrac{1}{2}y) - 1 \; dy$$

$$= \pi\left[2\tan(\tfrac{1}{2}y) - y\right]_0^{\frac{\pi}{2}}$$

$$= \pi\left[(2\tan\tfrac{\pi}{4} - \tan 0) - (\tfrac{\pi}{2} - 0)\right]$$

$\therefore$ Volume $= \frac{1}{2}\pi(4 - \pi)$ units^3.

1 for writing $x = \tan(\frac{1}{2}y)$

1 for definite integral

1 for working towards final answer

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c) For $n = 3$

$LHS = 3^3 + 4^3 \qquad RHS = 5^3$

$= 91 \qquad = 125$

$LHS < RHS \;\therefore$ true for $n = 3$

Assume true for $n = k$

ie $3^k + 4^k < 5^k$

ie $5^k > 3^k + 4^k$ —— (1)

Required to show true for $n = k+1$

ie $5^{k+1} > 3^{k+1} + 4^{k+1}$

Now, $LHS = 5^{k+1}$

$= 5 \times 5^k$

$> 5 \times (3^k + 4^k)$ substitution from (1)

$= 5 \times 3^k + 5 \times 4^k$

$> 3 \times 3^k + 4 \times 4^k$

$= 3^{k+1} + 4^{k+1}$

$= RHS$

ie $LHS > RHS$

~~∴~~ the result is true for $n = 3$ and proven true for $n = k+1$ after assuming true for $n = k$. $\therefore$ it is true for all positive integer $n \geq 3$.

1 for showing true for $n = 3$

1 for assumption

1 for showing true for $n = k+1$. Working must be shown.

Question 3

d) i) No restriction $= 9!$	1 for answer
ii) Number of ways girl is between 2 boys $= 2 \times 7!$	1 for answer
iii) 2 particular not wishing to be together $= 9! -$ them together $= 9! - 2\times 8!$ $= 282,240$	1 for subtracting $2\times 8!$ 1 for answer