Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2018

Year 12 Extension 1 Mathematics

30/11/18

General Instructions

- Reading time – 5 minutes
- Working time – 40 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

	Question 1	/5
	Question 2	/8
	Question 3	/3
	Question 4	/9
	Total	**/25**

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Section I

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 1 – 4, your response should include relevant mathematical reasoning and/or calculations.

Question 1 (5 marks) Use a new writing booklet.

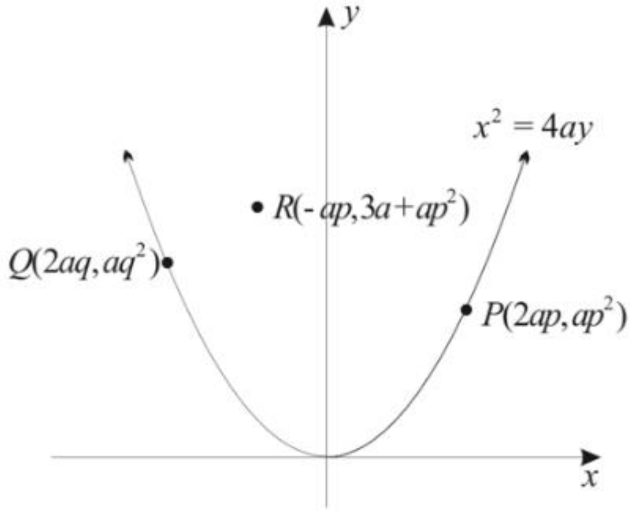
Consider the function $f(x) = (x-1)^3(x+1)$

(a) Find any stationary points of the function $f(x)$ and determine their nature. **3**

(b) Sketch $f(x)$ showing all major features. **2**

End of Question 1

Question 2 (8 marks) Use the a new writing booklet.



In the diagram above, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are distinct points on the parabola $x^2 = 4ay$, and R is the point $(-ap, 3a + ap^2)$.

(a) Show that the normal to the parabola at P has equation $x + py = 2ap + ap^3$. **2**

(b) Show that the normal at P passes through R. **1**

(c) If the normal at Q also passes through R, show that $q^2 + pq - 1 = 0$. **2**

(d) Show that there are always two real values of q satisfying the equation in part (c). **1**

(e) Deduce that three normals to the parabola, two of which are perpendicular to each other, pass through the point R. **2**
(You may assume that $p^2 = \frac{1}{2}$.)

End of Question 2

Question 3 (3 marks) Use the a new writing booklet.

A point R moves on the line $x + y = 1$ from $(1,0)$ to $(0,1)$. The point $P(x, y)$ moves such that the x-coordinate of P is the square of that of R and the y-coordinate is the same as that of R. **3**

Find the locus of P **and** draw the loci of both P and R.

End of Question 3

Question 4 (9 marks) Use the a new writing booklet.

Consider the function $f(x) = x^3 - 3Bx^2 + 24$, where B is an integer with a constant value.

(a) Show that the curve has stationary points at $x = 0$ and $x = 2B$. **1**

(b) Consider $B > 0$. **2**
Determine the nature of the stationary points at $x = 0$ and $x = 2B$.

(c) Let α be the **only** root of this function. **1**
Draw a possible curve with exactly one root, showing the turning points.

(d) Draw possible curves with exactly one root for $B < 0$ and for $B = 0$. **2**

(e) Explain why $\alpha < 0$. (That is, α is always negative) **1**

(f) For $B > 0$, explain why the only value B can take is $B = 1$. **2**

End of Question 4

Yr 12 - X1 Validation Answers

T4 - 2018

Start here

1 $\quad f(x) = (x-1)^3(x+1)$

$y = u \cdot v$

$u = (x-1)^3$

$u' = 3(x-1)^2$

$v = x+1$

$v' = 1$

$y' = u'v + v'u$

$y' = 3(x-1)^2 \cdot (x+1) + 1 \cdot (x-1)^3$ [1 mark]

$y' = (x-1)^2[3(x+1) + (x-1)]$

$y' = (x-1)^2[3x+3+x-1]$

$y' = (x-1)^2(4x+2)$

let $y' = 0$

$0 = (x-1)^2(4x+2)$

$\therefore$ stat pts at $x = 1$, $x = \frac{-1}{2}$ [1 mark]

x	-1	$-1/2$	0
y'	-8	0	2

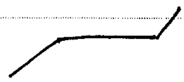
$\therefore$ min at $x = \frac{-1}{2}$

$y = \left(\frac{-1}{2} - 1\right)^3 \cdot \left(\frac{1}{2}\right)$

$y = \left(\frac{-3}{2}\right)^3 \cdot \frac{1}{2} = \frac{-27}{16}$

$\therefore \left(\frac{-1}{2}, \frac{-27}{16}\right)$

x	0	1	2
y'	2	0	10



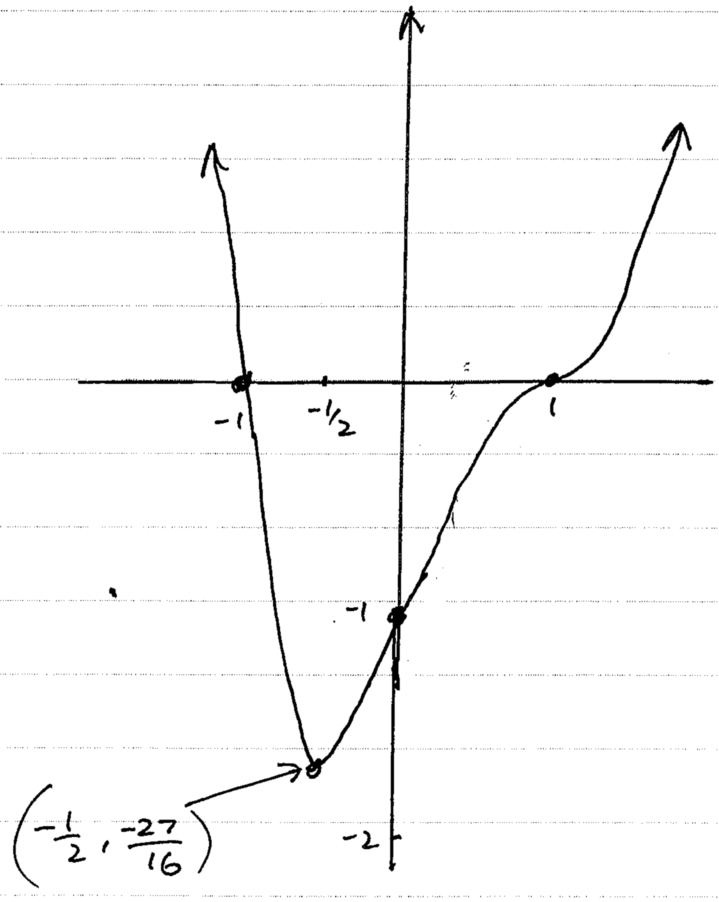
$\therefore$ Horizontal P.O.I at

$x = 1,\ y = 0$

$\therefore (1, 0)$

[gets both pts & correct nature - 1 mark]

b



Note

x-int's

at 1, -1

y-int at -1

[1 shape]
i.e. +ve quartic.

[1 - x, y - ints
& min
& horizontal
P.O.I]

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a) if done long way ... (<u>not</u> recommended!)

$f(x) = (x-1)^3(x+1)$

$= (x^2-1)(x-1)^2$

$= (x^2-1)(x^2-2x+1)$

$= x^4 - 2x^3 + x^2$

$\qquad - x^2 + 2x - 1$

$f(x) = x^4 - 2x^3 + 2x - 1$

$f'(x) = 4x^3 - 6x^2 + 2$

$= 2(2x^3 - 3x^2 + 1)$

let $x = 1$ $(x-1)$ is a factor

$x-1 \overline{)2x^3 - 3x^2 + 1}$

$\Rightarrow (x-1)(2x^2 - x - 1)$

$\Rightarrow (x-1)(x-1)(2x+1)$

$\therefore x = 1, -\frac{1}{2}$

note $f''(x) = 12x^2 - 12x$

$f''(1) = 12 - 12 = 0$ $\therefore$ possible P.O.I

test concavity

x	$\frac{1}{2}$	1	2
$f''(x)$	-3	0	24

$\therefore$ P.O.I at $(1, 0)$

$f''(-\frac{1}{2}) = 12 \times (-\frac{1}{2})^2 - 12 \times -\frac{1}{2}$

$= 9$ which is > 0 $\therefore$ min at $(-\frac{1}{2}, \frac{-27}{16})$

Tick this box if you have continued this answer in another writing booklet.

Office Use Only – Do NOT write anything, or make any marks, below this line.

Marker's Feedback

Q1

a
→ although the product rule was the easiest way, many students expanded poorly & lost marks
- some used the product rule incorrectly even though it is on the formulae sheet.
- a number didn't prove min, or horizontal P.O.I
- some didn't actually find the points.
  - students must answer the question...

→ Overall, this question was done at a far lower standard than the X1 group had shown in the 2 unit paper.

⇒ X1 students should always know & remember 2 unit content... Harder 2 unit Q's are X1 content!

b
Many didn't draw a quartic!!
- many didn't mark the x & y-intercepts & lost marks
- some didn't label the stationary points.
- this was sad given they had just done 2 assignments that involved drawing quartics...

Start here

2 a

$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$

$$y' = \frac{2x}{4a} \qquad (2ap, ap^2)$$

$$m_T = \frac{2 \times 2ap}{4a}$$

$$m_T = \frac{\cancel{4a}p}{\cancel{4a}} = p$$

$$m_N = \frac{-1}{m_T} = \frac{-1}{p} \quad \text{[1 mark]}$$

$$y - y_1 = m(x - x_1)$$
$$y - ap^2 = \frac{-1}{p}(x - 2ap)$$
$$py - ap^3 = -x + 2ap$$
$$\therefore x + py = 2ap + ap^3 \quad \text{[1 mark]}$$

b sub R into $x + py = 2ap + ap^3$

$R(-ap, 3a + ap^2)$

$$LHS = -ap + p(3a + ap^2)$$
$$= -ap + 3ap + ap^3$$
$$= 2ap + ap^3$$
$$= RHS \quad \text{[1 mark]}$$

$\therefore R$ lies on normal.

c from a normal at q is

$$x + qy = 2aq + aq^3$$

sub $R(-ap, 3a + ap^2)$

$$-ap + q(3a + ap^2) = 2aq + aq^3$$

$$(-ap + 3aq + ap^2 q = 2aq + aq^3) \div a$$

$$-p + 3q + p^2 q = \cancel{2q} + q^3$$

[1 mark] → to here or similar with 'a' cancelled out

$$q^3 = q + p^2 q - p$$

$$q^3 - p^2 q + p - q = 0$$

$$q(q^2 - p^2) + (p - q) = 0$$

as $q \neq p$. ← should show in the future!

$$q\cancel{(q-p)}(q+p) - \cancel{(q-p)} = 0$$

$$q(q+p) - 1 = 0$$

[1 mark proves result] $\therefore q^2 + pq - 1 = 0$

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d

$$q^2 + pq - 1 = 0$$

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or just use Δ

$$q = \frac{-p \pm \sqrt{p^2 - 4.1.-1}}{2}$$

$$q = \frac{-p \pm \sqrt{p^2 + 4}}{2}$$

[1 mark]

as $p^2 > 0$ and $4 > 0$

$\therefore$ always 2 real values of q as $\Delta > 0$

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e $q^2+pq-1=0$

roots are α, β

$\alpha\beta=-1$ [1 mark]

gradients of normals

$$\frac{-1}{\alpha} \times \frac{-1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{-1} = -1$$

$\therefore$ normals $\perp$ to each other.

in reality, α, β are q_1, q_2

$\therefore$ normals at $(2aq_1, aq_1^2)$

$(2aq_2, aq_2^2)$ are $\perp$ to each other.

from ii normal at P also goes through R.

[1 mark] $\therefore$ 3 normals pass through R.

Tick this box if you have continued this answer in another writing booklet.

Marker's Feedback

Q2

a was very well done overall
- some quoted the gradient and lost a mark.
In "show that" questions you need to show all steps.

b very well done

c poorly done overall.
About half the cohort tried to "fudge" the answer or went down the wrong path.
→ if students substituted R into the normal at Q and simplified, removing 'a' and collecting like terms they not only got a mark, but also put themselves in a position to complete the question.

d many students didn't realise the discriminant was required, and that they needed to use part c.
many either used it incorrectly or didn't conclude that their Δ is always > 0.
⇒ students should note that for $q^2 + pq - 1 = 0$

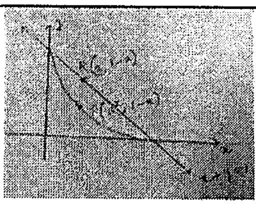
$$\Delta = b^2 - 4ac$$

$$b = p, \ a = 1, \ c = -1$$

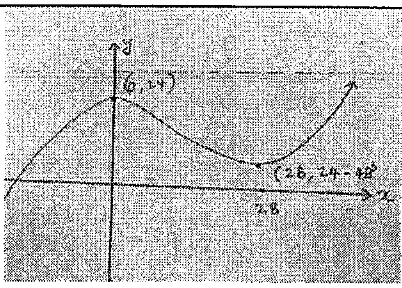
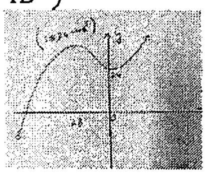
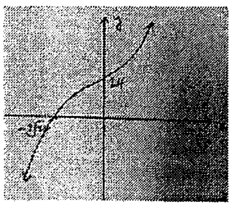
many thought $b = pq$, which doesn't make sense as p is the coefficient of q

e this was a very challenging question for the cohort:
The best way to tackle it, was to leverage part c and notice that the product of the roots was -1.
- many incorrectly assumed the normal at P $\perp$ to the normal at Q.
- an interesting question, well worth reviewing the answer & ensuring you understand the process.

Question 3

3.	 Any point on the line is of the form $R(a, 1-a)$ and hence $P(a^2, 1-a)$ Finding locus: $x = a^2$ and $y = 1 - a$ Hence, $a = \pm\sqrt{x}$ Hence, the locus of P is $y = 1 \pm \sqrt{x}$ Since, $0 \le a \le 1$, then $a^2 < a$ (the square of any number less than 1 is less than the number itself (See diagram) Hence, $y = 1 - \sqrt{x}$	3 marks: -Expresses the R and P in -coordinate form Plots -Draws the loci of R and P -Calculates with reasoning the locos of P 2 marks: Three of the criteria are demonstrated 1 mark Two of the criteria met	Many Students did not restrict the domain + range of the line $y = 1 - x$ Locus ~~for~~ P needed to be given algebraically + graphically. $x^2 = 1 - y$ is not sufficient, Justification was required to show why $y = 1 - \sqrt{x}$.

Question 4 This question was poorly done; with many poor solutions.

(a)	$f(x) = x^3 - 3Bx^2 + 24$ $f'(x) = 3x^2 - 6Bx$ For stationary points, $f'(x) = 0$. Hence, $3x^2 - 6Bx = 0$ $x(x - 2B) = 0$ $x = 0.\ and\ x = 2B$	1 mark: sets $f'(x) = 0$ and solves correctly to prove the result.	Need to state $f'(x)=0$ for stationary pts It is a "show that"
(b)	$f''(x) = 6x - 6B$ At $x = 0, f''(x) = -6B$ $B > 0$, hence $f''(0) < 0$ Concave down Hence maximum turning point at $(0, 24)$ At $x = 2B, f''(x) = 12B - 6B = 6B > 0$ as $B > 0$. Concave up and hence, Minimum turning point at $(2B, 24 - 4B^3)$	1 mark: determines the nature of stationary point at $(0, 0)$ 1 mark: Determines the nature of stationary point at $(2B, 24 - 4B^3)$	Students that used $f''(x)$ had more success Students need to state a max or min Not just concave up or down.
(c)		1 mark: correct shape and the turning points are labelled clearly.	Correct shape only was OK.
(d)	When $B < 0$, At $x = 0, f''(x) = -6B > 0$ Hence minimum turning point at $(0, 24)$ At $x = 2B, f''(x) = 12B - 6B = 6B < 0$ Maximum turning point at $(2B, 24 - 4B^3)$  When $B = 0, f(x) = x^3 + 24$ At $x = 0\ or\ x = 2B$, $f''(x) = -6B = 0$ 	1 mark each: tests the concavity to determine the max/min turning points and sketches the graph	Students did not use location of turning pts in Part a).
(e)	Referring to the graphs and the positions of turning points from (iii) and (iv), the x —intercept α is always negative for $B < 0$, $B = 0$ and $B > 0$	1 mark: refers to the graphs and the x- intercept	Students found this challenging if their graphs were incorrect.

(f)	When $B > 0$, referring to the values of the function ($y -$ values), $0 < 24 - 4B^2 < 24$ $-24 < -4B^2 < 0$ $0 < B^3 < 6$ $0 < B < \sqrt[3]{6} \approx 1.81 \ldots$ Hence, for $B > 0$, the only value of $B = 1$	1 mark: Use the turning points and gives the inequlity 1 mark: solves the inequality to prove the result	This part poorly done