Student Number

Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

2018 | HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1 Task 3

General Instructions

- Reading time – 5 minutes.
- Working time – 60 minutes.
- Write using blue or black pen
- NESA approved calculators may be used
- A formulae and data sheet is provided
- For questions 5 – 7, Show relevant mathematical reasoning and/or calculation

Section I – 4 marks (pages 2–3) • Attempt Questions 1 – 4 • Allow about 5 minutes for this section	Multiple Choice Questions	/4
Section II – 37 marks (pages 4 – 7) • Attempt Questions 5 – 7 • Allow about 55 minutes for this section	Question 5	/11
	Question 6	/13
	Question 7	/12
	Total	**/40**

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM
This assessment task constitutes 20% of the Higher School Certificate Course Assessment

Section I

4 marks
Attempt Questions 1 – 4
Allow about 5 – 7 minutes for this section

Use the multiple-choice answer sheet for questions 1 – 4 (Detach from paper)

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1 In how many different ways can the letters of the word 'OPTICAL' be arranged so that the vowels are always together?

(A) 610

(B) 720

(C) 825

(D) 920

2 At Hobbit school, there are six periods in each working day. In how many ways can one organise five subjects such that each subject is allowed at least one period?

(A) 1800

(B) 3200

(C) 3600

(D) 4150

3 The derivative of $y = x^2 \sin^{-1}(2x)$ is:

(A) $2xsin^{-1}(2x) + \frac{2x^2}{\sqrt{1-4x^2}}$

(B) $2xsin^{-1}(2x) + \frac{x^2}{\sqrt{1-4x^2}}$

(C) $2xsin^{-1}(2x) - \frac{4x^2}{\sqrt{1-4x^2}}$

(D) $2xsin^{-1}(2x) + \frac{x^2}{\sqrt{1-2x^2}}$

Section 1 continues on page 3

4 If a, b, c are consecutive terms of an arithmetic sequence as well as a geometric sequence, then:

(A) $a = b \neq c$

(B) $a \neq b = c$

(C) $a \neq b \neq c$

(D) $a = b = c$

End of Multiple choice section

Section II

36 marks
Attempt Questions 5 – 7
Allow about 55 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 5 – 7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (11 marks) Use a SEPARATE writing booklet

Marks

a) Find the derivative of $sin^{-1}(cosx)$ for $\pi < x < 2\pi$ **2**

b) Find the exact value of $cos(2\ tan^{-1}2)$ without the use of a calculator. **2**

c) The sum of the first ten terms of the series **2**

$$\log_2 \frac{1}{x} + \log_2 \frac{1}{x^2} + \log_2 \frac{1}{x^3} + \ \dots\dots \quad (x > 0) \ \text{ is } -440.$$

Find the value of x.

d) Consider the series

$$\sin^2 x + \sin^4 x + \sin^6 x + \ \dots\dots\dots\dots\dots \qquad 0 < x < \frac{\pi}{2}.$$

Show that the limiting sum exists **2**

e) A committee of five is to be formed from 4 Liberal senators, 3 Labor senators and 2 Democrat senators.

(i) How many different committees can be formed that have 3 Liberals, 1 Labor and 1 Democrat? **1**

(ii) If the committee is to be chosen at random, what is the probability that there will be a Liberal majority in the committee? **2**

End of Question 5

Question 6 (13 marks) Use a SEPARATE writing booklet

Marks

a) Eight people attend a meeting. They are provided with two circular tables, one seating 3 people and the other seating 5 people.

(i) How many seating arrangements are possible? **2**

(ii) If the seating is done randomly, what is the probability that a particular couple are on different tables? **2**

b) Andy played a game against the computer.

Andy gained 400 points on the first day he played. On each subsequent day, he gained 90% of the points he gained on the previous day.

Find the minimum number of days for Andy to gain at least 95% of the theoretical maximum points he can earn. **3**

c) Betty has one black, one yellow, two red, two green and three blue beads. The nine beads are arranged randomly in a line. Find the probability that:

(i) The two red beads are next to each other. **2**

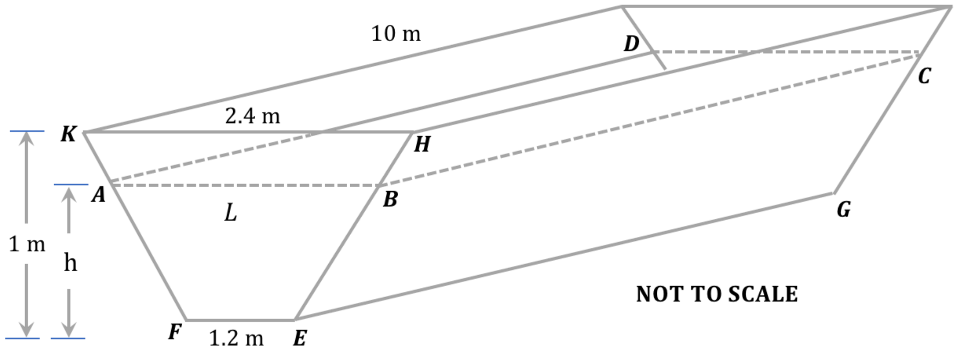
(ii) All red and green beads are separated. **2**

(iii) Either the two red beads are next to each other or the two green beads are next to each other or both. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet

Marks



a) The trough shown contains water to a depth of h metres. The volume of water inside the trough is given by $V = 6h(h + 2)$. (*Do not prove this*)

The trough is being topped up at a rate of $0.03m^3/min$.

(i) Find the rate at which the water level rises when the water is 0.6 m deep. Leave your answer in exact form. **2**

(ii) Find an expression for L, the length of AB. **2**

Hence show that the area of the water surface $ABCD$ is given by

$$S = 12h + 12$$

(iii) Find the rate at which the area of the water surface will increase when the water inside the trough is 0.6 metres deep. **2**

Question 7 continued on Page 7

Question 7 continued

b) Let $f(x)$ be a continuously differentiable function with the inverse function denoted by $g(x)$ and $f\big(g(x)\big) = x$. [Do not prove this].

(i) Show that **1**

$$g'(x) = \frac{1}{f'\big(g(x)\big)}$$

(ii) Given that $f(x) = \sqrt{x+2}$. **2**

Without finding $f^{-1}(x)$, find the derivative of $f^{-1}(x)$ using the result in part (i).

(iii) Find the domain of $f^{-1}(x)$. **1**

(iv) Find the points of intersection $f(x)$ and $f^{-1}(x)$ **2**

End of Paper

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Mathematics

Section I – Multiple Choice Answer Sheet

Use this multiple-choice answer sheet for questions 1 – 4. Detach this sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ☒ B ☒ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

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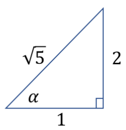
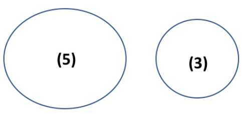
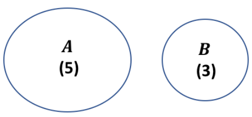
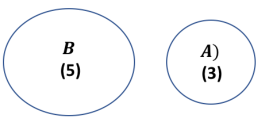
Marking scheme 2018 Task 3 Ext 1

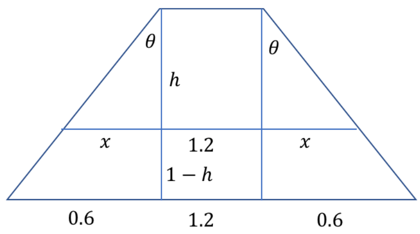
1.	3 vowels and 4 consonants No. of ways = $3! \times 5! = 720.$	B	
2	5 subjects can be organised in 6 subjects in 6_{P_5} Ways. Any of the 5 subjects can be organised in the remaining period in 5_{C_1} ways. Two subjects are alike in each of the arrangement. So, divide by 2!. Number of arrangements = $\frac{6_{P_5} \times 5_{P_1}}{2!} = 1800$	A	
3	$y = x^2 \sin^{-1}(2x)$ $\frac{dy}{dx} = x^2 . \frac{1 \times 2}{\sqrt{1-4x^2}} + 2x \sin^{-1}(2x)$ $= \frac{2x^2}{\sqrt{1-4x^2}} + 2x \sin^{-1}(2x)$	A	
4	$b = \frac{a+c}{2}$ $b^2 = ac$ $\left(\frac{a+c}{2}\right)^2 = ac$ $(a+c)^2 = 4ac$ $(a-c)^2 = 0$ $\therefore a = c. \quad \therefore b = \frac{c+c}{2} = c$ $a = b = c$	D	
Q5(a)	$\frac{d}{dx} \sin^{-1}(\cos x) = \frac{1}{\sqrt{1-\cos^2 x}} . (-sinx)$ $= \frac{1}{\sqrt{\sin^2 x}} . (-sinx)$. **1 mark** $= \frac{sinx}{\lvert sinx \rvert}$ =.1 for $\pi < x < 2\pi$. **1 mark**	1 mark: correct differentiation 1 mark: correct working and reasoning	
b)	Let $\alpha = \tan^{-1} 2$ $\therefore \tan \alpha = 2$  $sin\alpha = \frac{2}{\sqrt{5}}$ and $\cos \alpha = \frac{1}{\sqrt{5}}$ $\cos(2\tan^{-1} 2) = \cos^2 \alpha - \sin^2 \alpha$ $= \left(\frac{1}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = -\frac{3}{5}$	1 mark: calculates correct values of $sin\alpha$ and and $\cos \alpha$ 1 mark: calculates $\cos 2\theta$ using their $sin\alpha$ and and $\cos \alpha$	

c)	$\log_2 \frac{1}{x} + \log_2 \frac{1}{x^2} + \log_2 \frac{1}{x^3} + \ldots\ldots \quad (x > 0)$ $= -\log_2 x - 2\log_2 x - 3\log_2 x - \ldots\ldots$ $S_{10} = -440$ $= -\log_2 x\,(1 + 2 + 3 + \cdots + 10) = -440.$ **1 mark** $= -\log_2 x \left(\frac{10}{2}(1 + 10)\right) = -440$ $= -\log_2 x \times 55 = -440$ $= \log_2 x = 8$ $x = 2^8 = 256.$ **1 mark**	1 mark: produces the series 1 mark: calculates the sum of the series and hence, the value of x.	
d)	For limiting sum to exist, it needs to be proved that $\lvert r\rvert < 1$ $\lvert\sin^2 x\rvert < 1$ to be proved. For $0 < x < \frac{\pi}{2}$., $\quad 0 < \sin x < 1$ Hence, $0 < \sin^2 x < 1$ ie. $\lvert r\rvert < 1$ Limiting sum exists. (It is a convergent series)	1 mark: explains that for $0 < x < \frac{\pi}{2}$., $\quad 0 < \sin x < 1$ 1 mark: proves $\lvert r\rvert < 1$	
(e)(i)	No. of committees = $4_{C_3} \times 3_{C_1} \times 2_{C_1} = 24$	1 mark: correct answer with working	
(ii)	For Liberal majority, 4 Lib, 1 Lab $= 4_{C_4} \times 3_{C_1} = 3$ 4 Lib, 1 Dem $= 4_{C_4} \times 2_{C_1} = 2$ 3 Lib, 1 Lab, 1 Dem. = 24 from (i) 3 Lib, 2 Lab $= 4_{C_3} \times 3_{C_2} = 12$ 3Lib, 2 Dem $= 4_{C_3} \times 2_{C_2} = 4$ Total no. of committees = 3 + 2+24 + 12 + 4 = 45 P(Lib majority) = $\frac{45}{9_{C_5}} = \frac{5}{14}$	2 marks: correct answer from correct working 1 mark: gives either total number of committees correctly or calculates the correct probability	

Question 6			
(a)(i)	Choose 5 people for the first table, 8_{C_5} ways; arrange them around the table 4! Ways; then the rest of the 3 people can be arranged in 2! Ways.  Number of arrangements = $8_{C_5} \times 4! \times 2! = 2688$	2 marks: correct answer from correct working 1 mark: correct number of arrangements of 5 and 3 people around a circle	
(ii)	 or  Treat them as two groups: 6 people and 2 people (A, B) Choose one from (A and B) = 2 ways; Choose 4 from 6 = 6_{C_4} ways and arrange the five people around the table 4! Ways; the rest of the 3 can be arranged in 2! Ways.	2 marks: correct answer from correct working 1 mark: significant progress towards calculating the number of arrangements and calculates probability correctly	

	$\therefore$ total number of arrangements = $2 \times 6_{C_4} \times 4! \times 2!$ = 1440 $P(A \text{ and } B \text{ on separate tables}) = \frac{1440}{2688} = \frac{15}{28}$		
b)	The theoretical maximum = $\frac{a}{1-r} = \frac{400}{1-0.9} =$ 4000 points $S_n \geq 0.95 \times S_\infty$ $\frac{400(1-0.9^n)}{1-0.9} \geq 0.95 \times \frac{400}{1-0.9}$ 1 mark $1 - 0.9^n \geq 0.95.$ $0.9^n \leq 0.05$ $n \geq. \frac{\ln 0.05}{\ln 0.9} = 28.4 \ldots ....$ Minimum no. of days. =. 29. 1 mark (using their S_∞	1 mark: correctly calculates the theoretical maximum points 1 mark: writes the equation to represent the model 1 mark: solves the equation with working and gives the correct rounded answer	
c)(i)	Number of arrangements without restrictions = $\frac{9!}{2!2!3!}$ Required probability = $\frac{8!}{2!3!} \div \frac{9!}{2!2!3!} = \frac{2}{9}$ RR - - - - - - -	1 mark: Total no. of arrangements with no restriction 1 mark: Calculates the probability correctly (Working required)	
(ii)	No. of ways to slot in the red beads and the green beads = $6_{C_4} \times \frac{4!}{2!2!}$ $\therefore$ Required probability $= \frac{5!}{3!} \times 6_{C_4} \times \frac{4!}{2!2!} \div \frac{9!}{2!3!3!} = \frac{5}{42}$ OR – B – Y – BL – BL – BL – P(all R and G separated) = $\frac{5!}{3!} \times \frac{{}^6P_4}{2!2!} \div \frac{9!}{2!3!3!} = \frac{5}{42}$	2 marks: correct answer from correct working 1 mark: Significant progress towards calculating probability	
(iii)	P(2 red beads together and 2 green beads together) = $\frac{7!}{3!} \div \frac{9!}{2!\,2!\,3!} = \frac{1}{18}$ Required probability = $P(A) + P(B) - P(A \cap B)$ $= \frac{2}{9} + \frac{2}{9} - \frac{1}{18} = \frac{7}{18}$ OR P(2 R together, G not) = $\frac{6! \times 7_{C_2}}{3!} \div \frac{9!}{2!\,2!\,3!} = \frac{1}{6}$ P(2G together, R not) = $\frac{6! \times 7_{C_2}}{3!} \div \frac{9!}{2!\,2!\,3!} = \frac{1}{6}$ P(2 R together and 2 G beads together) = $\frac{1}{18}$ Total probability $= \frac{1}{6} + \frac{1}{6} + \frac{1}{18} = \frac{7}{18}$	1 mark: calculates the probability of arranging the red and the green beads together 1 mark: uses the answers in (i) for probabilities of 2 red beads together (same for 2 green beads together) and calculates the probability	

Question 7

a)(i)	$V = 6h(h+2)$ $\frac{dV}{dt} = .0.03\ m^3/min$ $V = 0.03\ t + C$ $V = 6h(h+2)$ $V = 6h^2 + 12h$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\frac{dV}{dh} = 12h + 12.$ $h = 0.6,\ \ \frac{dV}{dh} = 12 \times 0.6 + 12 = 19.2$ $\frac{dh}{dt} = \frac{0.03}{19.2} = 0.00156 \ldots ....$ $= \frac{1}{640}$ cm/min	1 mark: evaluates $\frac{dV}{dh}$ when h= 0.6 1 mark: calculates $\frac{dh}{dt}$	
(ii)	$h = 0,\ \ L = 1.2$ when $h = 1,\ \ L = 2.4$ $L = mh + b$ $L = mh + 1.2$ $2.4 = m + 1.2 \quad \therefore m = 1.2$ $L = 1.2h + 1.2.$ 1 mark $S = 10L = 10(1.2h + 1.2)$ $S = 12h + 12.$ 1 mark OR  $tan\theta = \frac{x}{h} = \frac{0.6}{1}$ $\therefore\ x = 0.6h$ $L = 1.2 + 2x$ $L = 1.2 + 2 \times 0.6$ $L = 1.2 + 1.2h$	1 mark: develops the expression for L 1 mark: expression for surface area OR 1 mark: correctly expresses x in terms of h 1 mark: Gives the correct expression for L	
(iii)	$S = 12h + 12$ $\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$ $=12 \times \frac{1}{640}$ $= \frac{3}{160}\ m^2/min$	1 mark: Calculates $\frac{ds}{dh}$ correctly 1 mark: differentiates and calculates $\frac{dS}{dt}$	
Q7(b) (i)	Given that $f(g(x)) = x$ $\therefore \frac{d}{dx} f(g(x)) = \frac{d}{dx}(x)$ $f'(g(x)).\frac{d}{dx}\ g(x) = 1$ $f'(g(x)).g'(x) = 1$ $\therefore g'(x) = \frac{1}{f' g(x)}$	1 mark: correct proof (all lines of working must be shown)	
(ii)	$f(x) = \sqrt{x+2}, \quad g(x) = f^{-1}(x)$ $f(g(x)) = x$ $f(f^{-1}(x)) = \sqrt{f^{-1}(x) + 2} = x.$ 1 mark $f'(x) = \frac{1}{2\sqrt{x+2}}$ From (i),	1 mark: identifies $\sqrt{f^{-1}(x) + 2} = x$	

	$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ $= \frac{1}{\frac{1}{2\sqrt{f^{-1}(x)+2}}}$ $= 2\sqrt{f^{-1}(x)+2}$ $= 2x.$ 1 mark	1 mark: proves $\frac{d}{dx} f^{-1}(x) = 2x$	
(iii)	Domain of $f^{-1}(x) =$ Range of $f(x)$ = All real x, $x \geq 0$	1 mark: correct answer	
(iv)	The point of intersection of $y = f(x)$and $y = f^{-1}(x)$ $y = x$ $x = \sqrt{x+2}$ 1 mark $x^2 = x + 2$ $x^2 - x - 2 = 0$ $(x+1)(x-2) = 0$ $x = -1$ or $x = 2$ Since, domain of $f^{-1}(x)$ is $x \geq 0$ $\therefore x = -1$ is rejected. $\therefore$ Point of intersection is $(2, 2)$. 1 mark	1 mark: demonstrates the understanding that $f(x)$and $f^{-1}(x)$ meet at $y = x$ 1 mark: solves for x and using the domain rejects $x = -1$, and gives the point of intersection	