Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

Q	Marks
MC	/3
4	/3
5	/3
6	/2
7	/6
8	/2
9	/4
10	/3
11	/3
12	/3
13	/3
14	/2
15	/3
16	/5
Total	/45

2019

Year 12

Extension 1

Task 1
Monday 2nd December

General Instructions:

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 45

Section I - 3 marks
- Allow about 5 minutes for this section

Section II - 42 marks
- Allow about 55 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

BLANK PAGE

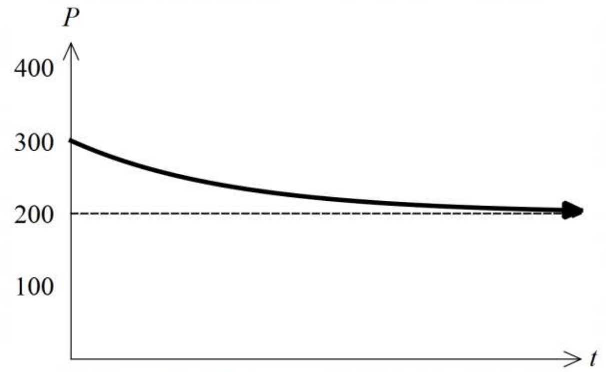
Section I

3 marks
Allow about 5 minutes for this section

Use the multiple-choice sheet for Question 1–3

---

1 The diagram shows the number of penguins, $P(t)$, on an island at time t.



Which equation best represents this graph?

(A) $P(t) = 200 - 100e^{-kt}$

(B) $P(t) = 200 + 100e^{-kt}$

(C) $P(t) = 300 - 100e^{-kt}$

(D) $P(t) = 300 + 100e^{-kt}$

2 Consider the following four statements. If there are more than m objects and there are m pigeonholes, then

1) There will be at least one pigeonhole with at least two objects,
2) There will be at least one pigeonhole with at least m objects,
3) There will be at least two pigeonholes with the same number of objects,
4) There will be at least two empty pigeonholes

Which of these are true?

(A) 1)

(B) 2)

(C) 3)

(D) 4)

3 Given that ${}^{n}C_7 + {}^{n}C_8 = {}^{11}C_8$ The value of n is:

(A) 8

(B) 9

(C) 10

(D) 11

End of Section I

Section II

In Questions 4 – 16, your response should include relevant mathematical reasoning and/or calculations.

Question 4 (3 marks)

(a) How many 3-digit numbers can be made from the digits 1, 2, 3, 4 and 5 without using any digit twice in the same number? **1**

(b) How many 3-digit numbers over 400 can be made without using any digit twice in the same number? **2**

Examination continues on next page.

Question 5 (3 marks)

A round table has 4 boys and 3 girls sitting around it.

(a) Find the number of ways of sitting possible if the boys and girls can sit anywhere around the table. **1**

(b) If the seating is arranged at random, find the probability that 2 particular girls will sit together. **2**

Question 6 (2 marks)

From six girls and five boys, a committee of seven is chosen. What proportion of all possible committees contain at least four girls? **2**

Examination continues on next page.

Question 7 (6 marks)

The population of sheep on a farm is given by $\frac{dN}{dt} = k(N - 1800)$, where N is the number of sheep

(a) Show that $N = 1800 + Ae^{kt}$ is the solution to the differential equation. **1**

(b) If there are initially 3000 sheep and after 3 years there are 3400 sheep, find:

(i) the number of sheep on the farm after 5 years. **3**

(ii) when the sheep population will reach 8000. **2**

Examination continues on next page.

Question 8 (2 marks)

Find the term independent of x in the expansion of **2**

$$\left(x^2 - \frac{1}{x}\right)^{12}$$

Question 9 (4 marks)

The binomial theorem states that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

(i) Show that **1**

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

(i) Hence, or otherwise, find the value of $\binom{50}{1} + \binom{50}{2} + \binom{50}{3} + \cdots \binom{50}{50}$ **1**

2

(i) Show that

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

.

Question 10 (3 marks)

If n is a positive integer, prove that $\left(2+\sqrt{13}\right)^n+\left(2-\sqrt{13}\right)^n$ is an integer. **3**

Question 11 (3 marks)

Prove by Mathematical Induction that **3**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots \ldots \ldots \ldots \ldots .. + n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

Question 12 (3 marks)

Prove by mathematical induction that $n(n+1)(n+2)$ is divisible by 6 for all positive integers n. **3**

Question 13 (3 marks)

Containers are coded by different arrangements of coloured dots in a row. The colours used are red, white and blue.

In an arrangement, at most three of the dots are red, at most two of the dots are white and at most one is blue.

(a) Find the number of different codes possible if six dots are used. **1**

(b) On some containers only five dots are used. Find the number of different codes possible in this case. Justify your answer. **2**

Question 14 (2 marks)

Six soccer teams play against each other and there were in total 16 soccer matches. **2**
Show that at least one pair of teams must have played against each other at least twice.

Question 15 (3 marks)

A committee of r people can be selected from a group consisting of n girls and two boys in ${}^{n+2}C_r$ different ways. From this, use the factorial formula to prove that **3**

$$ {}^{n+2}C_r = {}^{n}C_r + 2\,{}^{n}C_{r-1} + {}^{n}C_{r-2}. $$

Question 16 (5 marks)

The radius of a spherical balloon is expanding at a constant rate of 6cm/s. At what rate is the surface area of the balloon expanding if the rate of change of volume is $15cm^3/s$? **5**

End of examination.

Extra writing space

If you use this space, clearly indicate which question you are answering.

MATHEMATICS DEPARTMENT

Centre Number

A	N	S	W	E	R			

Student Number

2019

Year 12

Extension 1

Task 1

Monday 2nd December

Q	Marks
MC	/3
4	/3
5	/3
6	/2
7	/2
8	/4
9	/2
10	/3
11	/3
12	/3
13	/3
14	/3
15	/3
16	/3
17	/5
Total	/45

General Instructions:

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 45

Section I - 3 marks

- Allow about 5 minutes for this section

Section II - 42 marks

- Allow about 55 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

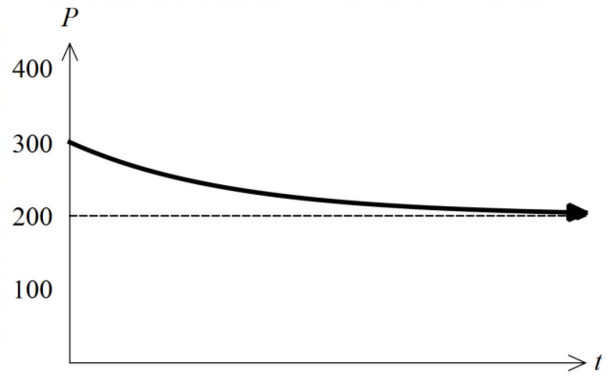
BLANK PAGE

Section I

3 marks
Allow about 5 minutes for this section

Use the multiple-choice sheet for Question 1–3

1 The diagram shows the number of penguins, $P(t)$, on an island at time t.



Which equation best represents this graph?

(A) $P(t) = 200 - 100e^{-kt}$

$P(t) = 200 + 100e^{-kt}$

(C) $P(t) = 300 - 100e^{-kt}$

(D) $P(t) = 300 + 100e^{-kt}$

2 Consider the following four statements. If there are more than m objects and there are m pigeonholes, then

1) There will be at least one pigeonhole with at least two objects,
2) There will be at least one pigeonhole with at least m objects,
3) There will be at least two pigeonholes with the same number of objects,
4) There will be at least two empty pigeonholes

Which of these are true?

1)

(B) 2)

(C) 3)

(D) 4)

3 Given that ${}^{n}C_7 + {}^{n}C_8 = {}^{11}C_8$ The value of n is:

(A) 8

(B) 9

10

(D) 11

End of Section I

Section II

In Questions 4 – 17, your response should include relevant mathematical reasoning and/or calculations.

Question 4 (3 marks)

(a) How many 3-digit numbers can be made from the digits 1, 2, 3, 4 and 5 without using any digit twice in the same number? **1**

$$^{5}P_{3} = 60$$

(b) How many 3-digit numbers over 400 can be made without using any digit twice in the same number? **2**

$$2 \times 4P_{2} = 24$$

Examination continues on next page.

Question 5 (3 marks)

A round table has 4 boys and 3 girls sitting around it.

(a) Find the number of ways of sitting possible if the boys and girls can sit anywhere around the table. **1**

$$\mathbf{6! = 720}$$

(b) If the seating is arranged at random, find the probability that 2 particular girls will sit together. **2**

$$\mathbf{2 \times \frac{5!}{720} = \frac{240}{720} = \frac{1}{3}}$$

Question 6 (2 marks)

From six girls and five boys, a committee of seven is chosen. What proportion of all possible committees contain at least four girls? **2**

$$\mathbf{\frac{6C_4 \times 5C_3 + 6C_5 \times 5C_2 + 6C_6 \times 5C_!}{11C_7} = \frac{215}{330} = \frac{43}{66}}$$

Question 7 (6 marks)

The population of sheep on a farm is given by $\frac{dN}{dt} = k(N - 1800)$, where N is the number of sheep

(a) Show that $N = 1800 + Ae^{kt}$ is the solution to the differential equation. **1**

$$\begin{aligned} \frac{dN}{dt} &= \boldsymbol{kAe^{kt}} \\ &= \boldsymbol{k(Ae^{kt} + 1800 - 1800)} \\ &= \boldsymbol{k(N - 1800)} \end{aligned}$$

(b) If there are initially 3000 sheep and after 3 years there are 3400 sheep, find:

(i) the number of sheep on the farm after 5 years. **2**

Sub. $t = 0 \quad and \quad N = 3000$

$$\boldsymbol{3000 = 1800 + Ae^{0} \quad \rightarrow \quad A = 1200}$$

$$\boldsymbol{3400 = 1800 + 1200e^{3k} \quad \rightarrow \quad k = \frac{ln\left(\frac{16}{12}\right)}{3} = 0.0959}$$

$$\therefore \quad \boldsymbol{N = 1800 + 1200e^{0.0959t}}$$

$$\boldsymbol{N = 1800 + 1200e^{5\times 0.0959} = 3738}$$

(ii) when the sheep population will reach 8000. **1**

$$\boldsymbol{8000 = 1800 + 1200e^{0.0959t} \quad \rightarrow \quad t = \frac{\ln\left(\frac{62}{12}\right)}{0.0959} = 17.1\ years}$$

Question 8 (2 marks)

Find the term independent of x in the expansion of **2**

$$\left(x^2 - \frac{1}{x}\right)^{12}$$

$$\boldsymbol{T_{k+1} = {}^{12}C_k\,(x^2)^{12-k}\left(-\frac{1}{x}\right)^k}$$

$$\boldsymbol{= {}^{12}C_k x^{24-2k}.\, x^{-k} \;=\; {}^{12}C_k x^{24-3k}}$$

Term independent of x when $\boldsymbol{24 - 3k = 0}$ → $\boldsymbol{k = 8}$

$$\therefore \quad \boldsymbol{T_9 = {}^{12}C_8 = 495}$$

Question 9 (4 marks)

The binomial theorem states that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

(i) Show that

$$2^n = \sum_{k=0}^{n}\binom{n}{k}$$

Using the binomial expansion from above, sub. $x = 1$

$$\boldsymbol{(1+1)^n = \binom{n}{0} + \binom{n}{1}1 + \binom{n}{2}1^2 + \cdots + \binom{n}{n}1^n.}$$

$$\rightarrow \quad \boldsymbol{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.}$$

$$\therefore \quad \boldsymbol{2^n = \sum_{k=0}^{n}\binom{n}{k}}$$

(i) Hence, or otherwise, find the value of $\binom{30}{1}+\binom{30}{2}+\binom{30}{3}+\cdots\binom{30}{50}$ **1**

$$\binom{30}{1}+\binom{30}{2}+\binom{30}{3}+\cdots\binom{30}{50} = \sum_{\boldsymbol{k=1}}^{\mathbf{30}}\binom{\mathbf{30}}{\boldsymbol{k}}-\binom{\mathbf{30}}{\mathbf{0}}$$

$$= \mathbf{2^{30} - 1 = 1073741823}$$

2

(i) Show that

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

.

Differentiate both sides of

$$\boldsymbol{(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n}.$$

$$\rightarrow \quad \boldsymbol{n(1+x)^{n-1} = \binom{n}{1}+2\binom{n}{2}x+3\binom{n}{2}x^2 \ldots + n\binom{n}{n}x^{n-1}}.$$

Sub. $\boldsymbol{x=1}$ $\rightarrow$ $\boldsymbol{n2^{n-1} = \binom{n}{1}+2\binom{n}{2}+3\binom{n}{2}\ldots+n\binom{n}{n}}$.

$$\therefore \quad \boldsymbol{n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}}$$

Question 10 (3 marks)

If n is a positive integer, prove that $\left(2+\sqrt{13}\right)^n+\left(2-\sqrt{13}\right)^n$ is an integer. **3**

$$\boldsymbol{\left(2+\sqrt{13}\right)^n = 2^n + nC_1 2^{n-1}\sqrt{13} + nC_2 2^{n-2}\times 13 \ldots\ldots\ldots + nC_n(\sqrt{13})^n} \qquad \textbf{(1)}$$

$$\boldsymbol{\left(2-\sqrt{13}\right)^n = 2^n - nC_1 2^{n-1}\sqrt{13} + nC_2 2^{n-2}\times 13 \ldots\ldots\ (-1)^n nC_n(\sqrt{13})^n} \qquad \textbf{(2)}$$

(1) + (2) $\boldsymbol{= 2^{n+1} + 2nC_2 2^{n-2}\times 13 + \cdots \ldots\ldots\ldots. + 2nC_n 13^{\frac{n}{2}}}$ **if n is even**

$\boldsymbol{= 2^{n+1} + 2nC_2 2^{n-2}\times 13 + \cdots \ldots\ldots\ldots. + 2nC_{n-1} 13^{\frac{n-1}{2}}}$ **if n is odd**

$\therefore$ **it is an integer (nC_r is integer)**

Question 11 (3 marks)

Prove by Mathematical Induction that **3**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots \ldots \ldots \ldots \ldots .. + n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$$

Step 1
Prove it is true for $n = 1$

LHS = $1.(1+1)^2 = 4$
RHS = $\frac{1}{12} 1.2.3.8 = 4 = RHS$ $\therefore$ **it's true for $n = 1$**

Step 2
Assume it's true for $n = k$

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots . + k(k+1)^2 = \frac{1}{12} k(k+1)(k+2)(3k+5)$$

Step 3
RTP it's true for $n = k + 1$
$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots . + (k+1)(k+2)^2$
$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$

By assumption (step 2) $+ (k+1)^{th}$ term ie $(k+1)(k+2)^2$ to both sides

RHS $= \frac{1}{12} k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$
$= \frac{1}{12}(k+1)(k+2)[k(3k+5) + 12(k+2)]$
$= \frac{1}{12}(k+1)(k+2)(3k^2 + 17k + 24)$
$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$ **= RHS of step 3**

$\therefore$ **It's true for $n = k + 1$**

It is true for $n = 1$. If it's true for $n = k$ then it's true for $n = k + 1$. Therefore, it's true for $n = 1, 2, 3, 4$ .... By MI it's true for all n.

Question 12 (3 marks)

Prove by mathematical induction that $n(n+1)(n+2)$ is divisible by 6 for all positive integers n. **3**

Step 1
Prove it's true for $n = 1$

LHS $1.2.3 = 6$ is divisible by 6
$\therefore$ it's true for $n = 1$

Step 2
Assume it's true for $n = k$
i.e $k(k+1)(k+2) = 6P$ for some integer P

Step 3
RTP it's true for $n = k + 1$

Ie proving $(k+1)(k+2)(k+3) = 6Q$ for some integer Q.

LHS = $k(k+1)(k+2) + 3(k+1)(k+2)$
$= 6P + 6 \times \frac{3}{k} = 6\left(P + \frac{3P}{k}\right)$ **Sub.** $(k+1)(k+2) = \frac{6P}{k}$ **from (2)**
$= 6Q$ **is divisible by 6, where** $Q = P + \frac{3P}{k}$,
and $\frac{3P}{k} = \frac{\frac{k}{2}(k+1)(k+2)}{k} = \frac{(k+1)(k+2)}{2}$ **is an integer.**

$\therefore$ It's true for $n = k + 1$

It is true for $n = 1$. If it's true for $n = k$ then it's true for $n = k + 1$. Therefore, it's true for $n = 1, 2, 3, 4$ .... By MI it's true for all n.

Question 13 (3 marks)

Containers are coded by different arrangements of coloured dots in a row. The colours used are red, white and blue.

In an arrangement, at most three of the dots are red, at most two of the dots are white and at most one is blue.

(a) Find the number of different codes possible if six dots are used. **1**

$$\frac{6!}{3!2!} = \mathbf{60}$$

(b) On some containers only five dots are used. Find the number of different codes possible in this case. Justify your answer. **2**

(3R & 2W), (2B,2W & 1B); (3R,1W & 1B)

$$\frac{5!}{3!\,2!} + \frac{5!}{2!\,2!} + \frac{5!}{3!} = \mathbf{60}$$

Question 14 (2 marks)

Six soccer teams play against each other and there were in total 16 soccer matches.
Show that at least one pair of teams must have played against each other at least twice.

Round	game play
1^{st}	5 games
2^{nd}	4 games
3^{rd}	3 games
4^{th}	2 games
5^{th}	1 games
Total	15 games

Or $^{6}C_2 = 15$ games

No. of pigeon holes is 15

No. of pigeons is 16

$$\frac{16}{15} = 1\ r\ 1$$

∴ at least one pair of teams must have played against each other at least twice

Question 15 (3 marks)

A committee of r people can be selected from a group consisting of n girls and two boys in ${}^{n+2}C_r$ different ways. From this use the factorial formula to prove that **3**

$$ {}^{n+2}C_r = {}^{n}C_r + 2\,{}^{n}C_{r-1} + {}^{n}C_{r-2}. $$

RHS $= \dfrac{n!}{(n-r)!r!} + \dfrac{2.n!}{(n-r+1)!(r-1)!} + \dfrac{n!}{(r-2)!(n-r+2)!}$ 1 mark-correct formula

$= \dfrac{(n-r+2)(n-r+1)n!+2n!r(n-r+2)+n!.r.(r-1)}{(n-r+2)!r!}$ 1 mark-correct expression with C.D.

$= \dfrac{n!((n-r+2)(n-r+1)-2nr-2r^2+4r+r^2-r}{(n-r+2)!r!}$

$= \dfrac{n!(n^2-nr+n-nr+r^2-r+2n-2r+2+2nr-2r^2+4r+r^2-r)}{(n-r+2)!r!}$

1 mark (showing expansion & simplified answer)

$= \dfrac{n!(n^2+3n+2)}{(n-r+2)!r!}$

$= \dfrac{(n+2)(n+1)n!}{(n-r+2)!r!} = \dfrac{(n+2)!}{(n-r+2)!r!}$

LHS $= \dfrac{(n+2)!}{r!(n-r+2)!} = RHS$

Question 16 (5 marks)

The radius of a spherical balloon is expanding at a constant rate of 6cm/s. At what rate is the surface area of the balloon expanding if the rate of change of volume is 15cm^3/s?

$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2$

$A = 4\pi r^2 \qquad \frac{dA}{dr} = 8\pi r$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 6 = 24\pi r^2$ **sub.** $\frac{dv}{dt} = 15$

$\rightarrow \quad 24\pi r^2 = 15 \quad \therefore \quad r = \sqrt{\frac{5}{8\pi}}$ **(2 marks-finding correct r)** **2**

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times 6 = 48\pi r$ **Sub** $r = \sqrt{\frac{5}{8\pi}}$ **(2 marks-finding correct $\frac{dV}{dt}$)** **2**

$\rightarrow \frac{dA}{dt} = 48\pi \times \sqrt{\frac{5}{8\pi}} = \sqrt{1440\pi}\ cm^2/s$ **(1 mark correct ans. $\frac{dA}{dt}$)** **1**

$= 67.3\ cm^2/s$