
Centre Number

Student Number

Killara HIGH SCHOOL

MATHEMATICS DEPARTMENT

2019

Year 12 Extension 1 Mathematics

Task 3 18th June 2019

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 44

Section I – 2 marks

- Allow about 3 minutes for this section

Section II – 39 marks

- Allow about 57 minutes for this section

Section I (2 marks)	Multiple Choice	/2
Section II (39 marks)	Question 3	/10
	Question 4	/11
	Question 5	/9
	Question 6	/9
	Total	**/41**

This question paper must not be removed from the examination room.
This assessment task constitutes 25% of the course.

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Question 1

The exact value of $\tan^{-1}(-\frac{1}{\sqrt{3}})$ is:

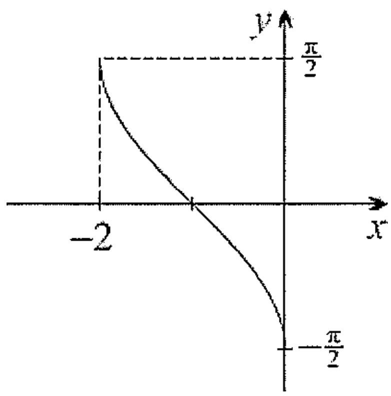
(A) $\frac{5\pi}{6}$

(B) $-\frac{5\pi}{6}$

(C) $-\frac{\pi}{6}$

(D) $\frac{\pi}{3}$

Question 2



The diagram above shows the graph of

(A) $y = \sin^{-1}(x+1)$

(B) $y = \sin^{-1}(x-1)$

(C) $y = \cos^{-1}(x+1) - \frac{\pi}{2}$

(D) $y = \cos^{-1}(x-1) - \frac{\pi}{2}$

Question 3 (10 marks) Use a **SEPARATE** writing booklet

(a) Find the derivative of $y = \cos^{-1}(2x)$ **1**

(b) Evaluate $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(c) State the domain and range of $y = 3\cos^{-1}\frac{x}{2}$? **2**

(d) Find the exact value of $\cos\left[\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{4}{3}\right)\right]$ **2**

(e)

i) Show that $\frac{d}{dx}\left(xcos^{-1}x - \sqrt{1-x^2}\right) = \cos^{-1}x$ **2**

ii) Hence evaluate $\int_0^1 \cos^{-1}x\,dx$ **1**

Question 4 (11 marks) Use a **SEPARATE** writing booklet

(a) How many arrangements of the word "excellent" are possible:

i) if there are no restrictions? **2**

ii) the word starts and end with an "l" ? **1**

iii) the three "e" 's are together ? **2**

(b) A committee of five, is to be chosen from 7 New South Welshman and 6 Queenslanders:

i) How many committee's can be chosen? **1**

ii) What is the probability that the committee has 5 Queenslanders? **1**

(c) 8 men and 8 women are to be seated at a round table.

i) In how many ways can they be seated if men and women alternate? **2**

ii) A particular man and a woman refuse to sit next to each other. How many ways can the group now be seated at the round table, if men and women alternate? **2**

Question 5 (9 marks) Use a **SEPARATE** writing booklet

(a) Prove ${}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$ **3**

(Note: this proof assumes that $1 \le k \le n-1$)

(b) Prove by Induction that the sum of the first n powers of 3 (starting with $n = 0$) is given by: **3**

$3^0 + 3^1 + 3^2 + ... + 3^n = \frac{1}{2}\left(3^{n+1} - 1\right)$

(c) Prove by Induction that $3^{2n} - 1$ is divisible by 8, for all $n \ge 1$ **3**

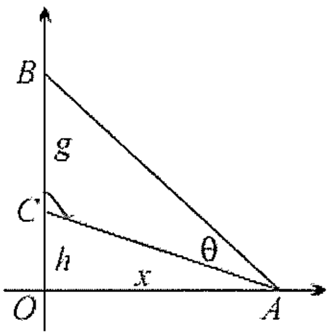
Question 6 (9 marks) Use a SEPARATE writing booklet

(a) Let $f(x) = 3 + 2x - x^2$. For $x \leq 1$ this function has an inverse $f^{-1}(x)$.

i) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same number plane. **2**

ii) Find an expression for $f^{-1}(x)$. **2**

(b)



A billboard of height g metres is mounted on a wall at BC which is h metres above level ground at O. The billboard subtends an angle θ when observed at ground level at A, x metres from O. $\angle BOA$ is a right angle.

i) Show that $\theta = \tan^{-1}\left(\frac{g+h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$. **1**

ii) Show that θ is a maximum when $x = \sqrt{h(g+h)}$. **2**

iii) It is known that the maximum value of θ is $\frac{\pi}{4}$, show that $g = 2h(1 + \sqrt{2})$. **2**

End of Examination

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2019-Assessment 2 Ex1 Jake, Q3

1a

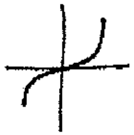
$y = \tan^{-1} x$ for $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

$\therefore \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

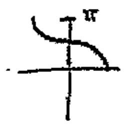
$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{-\pi}{6}$ $\therefore$ c

2 $\sin^{-1} x$



$\cos^{-1} x$



$\therefore \cos^{-1} x$ variant

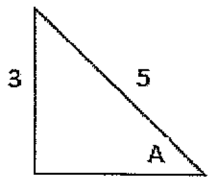
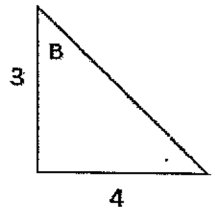
centre of graph shifted to $x = -1$

$\therefore \cos^{-1}(x+1) \frac{-\pi}{2}$

$\therefore$ c

Question 3

(a) $\frac{dy}{dx} = -\frac{2}{\sqrt{1-4x^2}}$ MUST USE REFERENCE SHEET	1-correct answer	
b) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\cancel{\tfrac{1}{2}}\sin^{-1}\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$	1-correct integration 1-correct answer	

c) Domain: $-1 \le \frac{x}{2} \le 1$ $-2 \le x \le 2$ Range: $0 \le y \le 3\pi$	2-correct domain & range	
d) $\cos\left[\sin^{-1}\left(\frac{3}{5}\right) - \tan^{1}\left(\frac{4}{3}\right)\right]$ Let $A = \sin^{-1}\left(\frac{3}{5}\right)$ $B = \tan^{1}\left(\frac{4}{3}\right)$ 	1-correct expressions of triangles	
$\cos(A - B) = cosA.cosB + sinA.sinB$ $= \frac{4}{5}.\frac{3}{5} + \frac{3}{5}.\frac{4}{5} = \frac{24}{25}$	1-correct answer with working	
e) (i) $\frac{d}{dx}\left(xcos^{-1}x - \sqrt{1 - x^2}\right) = 1.\cos^{-1}x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2} \times -\frac{2x}{\sqrt{1-x^2}}$ $= \cos^{-1}x$	2-correct derivative of both functions with working	
(ii) $\int_0^1 \cos^{-1}x\,dx = \left[xcos^{-1}x - \sqrt{1 - x^2}\right]_0^1 = 1$	1-correct answer	

$-\frac{2}{\sqrt{1-4x^2}}$

Question 4

a i $= \frac{9!}{3!2!} = 30,240$	2 marks: Correct answer. 1 mark: any one of the repetition is missing	Well done.
ii $\boxed{\ell}\square\square\square\square\square\square\square\boxed{\ell}$ $= \frac{7!}{3!} = 840$ (have to account for 3×e's)	Correct answer with correct working 1 mark.	Well done.
iii $\boxed{3\times e's}\square\square\square\square\square\square$ $= \frac{7!}{2!}$ (have to account for 2×ℓ's) $= 2520$	2 marks: Correct answer from correct working 1 mark: $\frac{7!}{3!2!}$ (minor error).	Generally well done Some students did this error.
b 7 NSW, 6 QLD = 13 in total, committee of 5 i $^{13}C_5 = 1287$	1 mark	Well done.
ii to pick 5 Q'landers from 6 = $^{6}C_5$ $\therefore P(5 \text{ Q'landers}) = \frac{^{6}C_5}{^{13}C_5} = \frac{6}{1287} = \frac{2}{429}$	1 mark (working required)	Well done.

4c i) 8m, 8w, round table  imagine locking in 1man 7m left in 7! ways 8w left in 8! ways $\therefore$ $7! \times 8!$ as if we lock a woman in 1st, we get the <u>same</u> arrange $\therefore$ $7! \times 8! = 203{,}212{,}800$	2 marks: Correct answer from Correct working $7! \times 7!$ received 1 mark.	Many students fixed one person and got $1 \times 7!$ correct. However, they forgot the 8 women can take any of the 8 positions, thus $8!$

NOTE: • explanation is important to get part marks.

ii. ← lock in particular man $\boxed{W}$ $\boxed{M}$ $\boxed{W}$ choose 2 women from 7 that aren't the "particular woman" and arrange 2 ways 7 man left in 7! ways 6 women left in 6! ways $\therefore 2 \times {}^7C_2 \times 7! \times 6! = 76{,}204{,}800 \times 2$ $= 152{,}409{,}600$ OR lock (M, W) together $= 2 \times 7! \times 7! = 50{,}803{,}200$ $\therefore$ not together $= 203{,}212{,}800 - 50{,}803{,}200$ $= 152{,}409{,}600$	2 Marks : Correct answer from correct working Ans (i) − 2×7!×7! 1 mark : Subtracting incorrect 2×7!×7! from answer (i)	Students had extreme difficulty with this question

Question 5

using $^nC_r = \frac{n!}{(n-r)!\,r!}$

$\therefore$ LHS $= \frac{n!}{(n-k)!\,k!}$

RHS $= \frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!}$

$= \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}$ [1 mark]

need common denom'r of $(n-k)!\,k!$

$= \frac{(n-1)!}{(n-k)!\,(k-1)!} \times \frac{k}{k} + \frac{(n-1)!}{(n-k-1)!\,k!} \times \frac{(n-k)}{(n-k)}$ [1 mark]

$= \frac{k(n-1)! + n \times (n-1)! - k(n-1)!}{(n-k)!\,k!}$

$= \frac{n!}{(n-k)!\,k!} =$ LHS [1 mark]

5b Prove $3^0 + 3^1 + 3^2 + \dots + 3^n = \frac{1}{2}\left(3^{n+1} - 1\right)$

when $n = 0$

$3^0 = 1 = LHS$

$RHS = \frac{1}{2}\left(3^{0+1} - 1\right) = \frac{1}{2}(3-1) = \frac{1}{2} \cdot 2 = 1 = LHS$:)

assume true for $n = k$

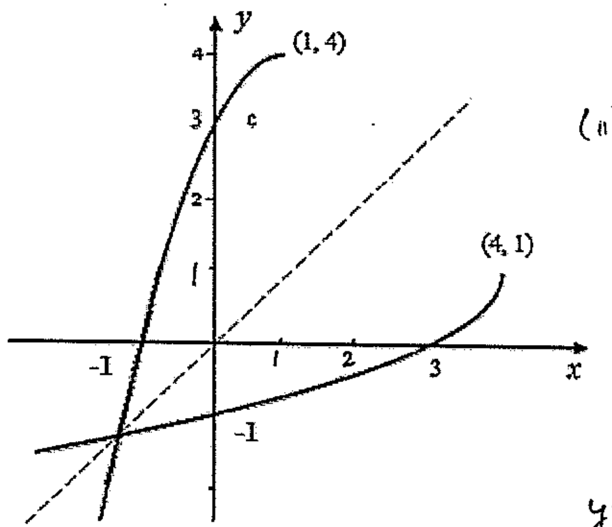
$$3^0 + 3^1 + 3^2 + \dots + 3^k = \frac{1}{2}\left(3^{k+1} - 1\right)$$

prove true for $n = k+1$

$$\underbrace{3^0 + 3^1 + 3^2 + \dots + 3^k}_{\text{using expression for } n=k} + 3^{k+1} = \frac{1}{2}\left[3^{(k+1+1)} - 1\right]$$

Question 6

(a) (i)



(ii)

$$x = 3 + 2y - y^2$$

$$y^2 - 2y - (3 - x) = 0$$

$$y = \frac{2 \pm \sqrt{4 + 4(3 - x)}}{2}$$

$$= \frac{2 \pm \sqrt{16 - 4x}}{2} = 1 \pm \sqrt{4 - x}$$

$$\therefore\ y = 1 - \sqrt{4 - x} \qquad y \le 1$$

or

$$y^2 - 2y = 3 - x$$

$$y^2 - 2y + 1 = 4 - x$$

$$(y - 1)^2 = 4 - x$$

$$y = 1 \pm \sqrt{4 - x}$$

1-correct shape

1-correct labelling

2- correct answer with working

1 mark: $1 \pm \sqrt{4 - x}$.

(i) When sketching graphs, you must show scales and labels on axes. Label major features. (x, y) changed to (y, x) on the inverse must be clearly shown. Put the $y = x$ line in.

Students continue to have difficulty using quadratic formula & completing the square

(b)

$$\theta = \angle OAB - \angle OAC$$

$$\tan \angle OAB = \frac{g+h}{x} \quad \rightarrow \quad \angle OAB = \tan^{-1}\left(\frac{g+h}{x}\right)$$

$$\tan \angle OAC = \frac{h}{x} \quad \rightarrow \quad \angle OCA = \tan^{-1}\left(\frac{h}{x}\right)$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{g+h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right) \qquad (1) \quad 1 \text{ mark.}$$

$$\frac{d\theta}{dx} = \frac{1}{1 + \frac{(g+h)^2}{x^2}} \times -\frac{g+h}{x^2} + \frac{1}{1 + \frac{h^2}{x^2}} = -\frac{g+h}{x^2 + (g+h)^2} + \frac{h}{x^2 + h^2}$$

For max. θ, $\quad \frac{d\theta}{dx} = 0 \quad \rightarrow \quad -\frac{g+h}{x^2 + (g+h)^2} + \frac{h}{x^2 + h^2} = 0$

$$-(g + h)(x^2 + h^2) + h(x^2 + (g + h)^2) = 0$$

$$-gx^2 - \cancel{gh^2} - \cancel{hx^2} - \cancel{h^3} + \cancel{hx^2} + hg^2 + \cancel{2}gh^2 + \cancel{h^3} = 0$$

$$-gx^2 + hg^2 + gh^2 = 0$$

$$\therefore x^2 = \frac{hg(g+h)}{g} \quad \rightarrow \quad x = \sqrt{(h(g + h)} \qquad (x > 0) \quad 1 \text{ mark.}$$

$$\frac{d^2\theta}{dx^2} = -\frac{2x(g + h)}{(x^2 + (g + h)^2)^2} - \frac{2xh}{(x^2 + h^2)^2} < 0 \quad \therefore \quad \theta \text{ is a maximum.} \qquad 1 \text{ mark.}$$

1-correct answer with full working shown

1-correct derivative

1-correct answer with working

- Most students received 1 marks for finding the expression for θ.
- Some students experienced difficulty differentiating $\frac{g+h}{x}$, & $\frac{h}{x}$.
- Very few students proved the x-value was a maximum. $\frac{dx}{d\theta} = 0$ gives you stationary point

Sub. $x=\sqrt{(h(g+h)}$ and $\theta=\frac{\pi}{4}$ into (1) $\frac{\pi}{4}=\tan^{-1}\left(\frac{g+h}{\sqrt{h(g+h)}}\right)-\tan^{-1}\left(\frac{h}{\sqrt{h(g+h)}}\right)$ $\rightarrow\quad 1=\frac{\frac{g+h}{\sqrt{h(g+h)}}-\frac{h}{\sqrt{h(g+h)}}}{1+\frac{h(g+h)}{h(g+h)}}\quad\rightarrow\quad\sqrt{\frac{g+h}{h}}-\sqrt{\frac{h}{(g+h)}}=2$ square both sises $\rightarrow\quad\frac{g+h}{h}-2+\frac{h}{g+h}=4\quad\rightarrow\quad\frac{g+h}{h}-\frac{h}{g+h}=6$ $(g+h)^2+h^2=6h(g+h)$ completing the square $(g+h)^2-6h(g+h)+9h^2=8h^2$ $\left((g+h)-3h\right)^2=8h^2\quad\rightarrow\quad(g-2h)^2=8h^2$ $g-2h=2\sqrt{2}h\quad\rightarrow\quad g=2h(1+\sqrt{2})$	1-showing the substitution of x & θ. 1-correct answer with working	Most Students received the first mark.

5b Prove $3^0 + 3^1 + 3^2 + \dots + 3^n = \frac{1}{2}(3^{n+1} - 1)$

when $n = 0$

$3^0 = 1 = LHS$

$RHS = \frac{1}{2}(3^{0+1} - 1) = \frac{1}{2}(3-1) = \frac{1}{2} \cdot 2 = 1 = LHS$ ∴ [1 mark]

assume true for $n = k$

$$3^0 + 3^1 + 3^2 + \dots + 3^k = \frac{1}{2}(3^{k+1} - 1)$$

prove true for $n = k+1$

$$\underbrace{3^0 + 3^1 + 3^2 + \dots + 3^k}_{\text{using expression for } n=k} + 3^{k+1} = \frac{1}{2}\left[3^{(k+1+1)} - 1\right]$$

$$\frac{1}{2}(3^{k+1} - 1) + 3^{k+1} = \frac{1}{2}(3^{k+2} - 1) \quad \text{[1 mark]}$$

with LHS

$$= \frac{1}{2}(3^{k+1} - 1) + \frac{2 \cdot 3^{k+1}}{2}$$

$$= \frac{3 \cdot 3^{k+1} - 1}{2}$$

$$= \frac{1}{2}(3^{k+2} - 1) = RHS$$

∴ as true for $n = 0$, assumed true for $n = k$, and proven true for $n = k+1$

∴ By induction it must be true for $n = 0, 1, 2 \dots$

∴ true for all $n \geq 0$.

[1 mark

5c Prove $3^{2n}-1$ divisible by 8 for $n \geq 1$

when $n=1$

$3^{2\times1}-1 = 3^2-1 = 9-1 = 8$

$8 \div 8 = 1$ $\therefore$ true for $n=1$ [1 mark]

assume true for $n=k$

$3^{2k}-1 = 8a$ (where a is an integer >0)

prove true for $n=k+1$

$3^{2(k+1)}-1 = 8b$ (where b is a positive integer)

$LHS = 3^{2k+2}-1$

$= 3^{2k} \cdot 3^2 - 1$

from $n=k$ assumption

$3^{2k} = 8a+1$

sub into LHS

$= (8a+1)9-1$ [1 mark]

$= 72a+9-1$

$= 72a+8$

$= 8(1+9a) = 8b$

$\therefore$ true for $n=k+1$

$\therefore$ as true for $n=1$, assumed true for $n=k$, and proven true for $n=k+1$, by induction it is true for $n=2,3$, etc ... $\therefore$ true for all $n \geq 1$ [1 mark]

Q5 Exam Feedback

a many students had forgotten ${}^{n}C_{k} = \frac{n!}{(n-k)!\,k!}$

- some tried to prove by induction & got in a mess.
- many didn't get a common denominator, or got a HUGE common denominator that made the question more difficult.
- some skipped lines of working & lost a mark.
- most students need to revise.

b was quite well done overall.

- some tested $n=1$, not $n=0$
- most did $n=0$, and substituted $n=k$ expression correctly, some need to work on Algebra skills / index skills as couldn't complete the question

c mostly very well done.

- some forgot to have a different pronumeral for 8 times e.g $8a$, $8b$ or $8Q$, $8M$ at $n=k$, $n=k+1$ stages.
- most got full marks
- some silly arithmetic / manipulation errors.

Note: for part b, c $\Rightarrow$ some students didn't conclude the proof correctly and a mark was deducted.