Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2020 **YEAR 12**

Mathematics Extension 1

Task 1

26th November, Thursday

General Instructions:

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 40

Section I - 3 marks

- Allow about 5 minutes for this section

Section II - 37 marks

- Allow about 45 minutes for this section

Q	Marks
MC	/3
4	/8
5	/9
6	/10
7	/10
Total	/40

This question paper must not be removed from the examination room.

This assessment task constitutes 20% of the course.

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Section I

3 marks
Allow about 5 minutes for this section

Use the multiple-choice sheet for Question 1–3

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1 A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. The expression which represents the number of ways this can be done is:

(A) ${}^{8}\mathbf{P}_{3} \times {}^{6}\mathbf{P}_{4}$

(B) ${}^{8}\mathbf{P}_{3} + {}^{6}\mathbf{P}_{4}$

(C) ${}^{8}\mathbf{C}_{3} \times {}^{6}\mathbf{C}_{4}$

(D) ${}^{8}\mathbf{C}_{3} + {}^{6}\mathbf{C}_{4}$

2 What is the coefficient of x^3 in $(2+x)^5$?

(A) 10

(B) 20

(C) 40

(D) 80

3 A drawer contains 12 red socks and 12 blue socks, all unmatched. How many socks must be drawn to ensure that you have at least 2 blue socks?

(A) 3

(B) 4

(C) 13

(D) 14

End of Section I

Section II

37 marks
Attempt Questions 4–7
Allow about 45 minutes for this section

Answer each question in a NEW booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 4 (8 marks)

(a) How many distinct eight letter arrangements can be made using the letters of the word PARALLEL? **1**

(b) If ${}^{2n}\mathbf{P}_{n-1} = 10 \times {}^{2n-1}\mathbf{P}_{n-2}$, find the value of n. **2**

(c) Ten people are attending a dinner party. The ten people will be seated at a round table.

(i) In how many ways can the ten people be arranged around the table? **1**

(ii) Two guests, Mark and Cecil wish to sit next to each other. In how many ways can the ten people be arranged such that Mark and Cecil are sitting next to each other? **1**

(d) The amount P of a medication in a patient, after being injected, decreases at a rate proportional to the amount present.

The initial amount of medication injected into the patient is 2 mL and the hourly rate of decrease is 3% of the drug present, at the beginning of that hourly period.

(i) How much of the drug, correct to two decimal places, still remains in the body after 5 hours? **1**

(ii) Find the rate of decrease of the drug when $t = 5$, correct to two significant figures. **1**

(e) In how many ways can you spell the word NOON in the grid below? **1**

You can start on any letter then on each step move one step in any direction (left, right, up, down, or diagonally).

You can return to the same 'N' from which you started.

N	**N**	**N**	**N**
N	**O**	**O**	**N**
N	**O**	**O**	**N**
N	**N**	**N**	**N**

Examination continues on next page

Question 5 (9 marks)

Answer each question in a NEW booklet.

(a) Find the coefficient of x^8 in the expansion $\left(\frac{2}{x} + x^3\right)^{20}$. **2**

(b) Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$. **2**

(c) A girls' volleyball team has 14 players, including a set of triplets: Amanda, Amy and Anna. **1**

In how many ways can we choose 6 starters if at least one of the triplets must be in the starting line-up?

(d) During summer, the population of fruit flies, P, is observed. The rate of increase of the population per week is modelled by the equation

$$\frac{dP}{dt} = k(P - 3000)$$

where k is a constant. At the beginning of summer, the population is 4000 and after 4 weeks, it is 10 000.

(i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, where A is a constant and t is the time in weeks. **1**

(ii) Find the values of A and k. **2**

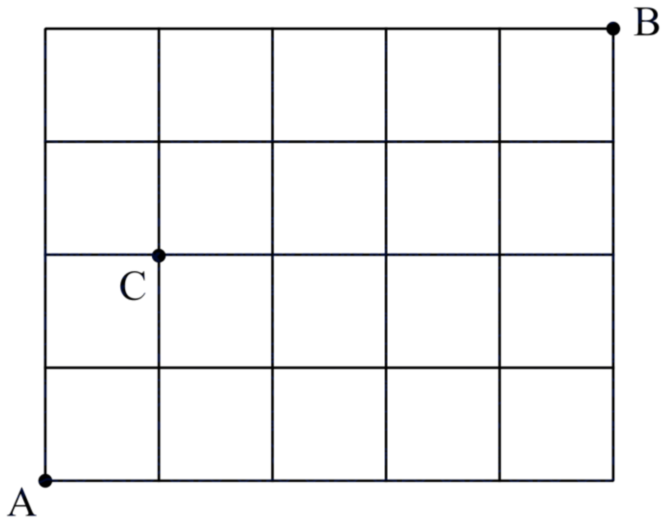
(iii) After how many weeks does the population reach half a million? Give your answer correct to 1 decimal place. **1**

Question 6 (10 marks)

Answer each question in a NEW booklet.

(a) The radius of a circular spill (r km), at a time t hours after it was first observed is given by $r = \dfrac{1 + 3t}{1 + t}$. Find the exact rate of increase of the area of the oil spill when the radius is 2 kilometres. **3**

(b) A nine-step path is drawn from A to B in the diagram below. If it is drawn randomly what is the probability it passes through C? **2**



(c) The motion of a particle satisfies the displacement-time equation $x = 4te^{-2t}$. **5**

Describe its motion with reference to:

- When the particle changes direction and where
- The displacement as $t \to \infty$
- The velocity as $t \to \infty$
- Whether the particle speeds up or slows down as $t \to \infty$

Examination continues on next page

Question 7 (10 marks)

Answer each question in a NEW booklet.

(a) (i) Using the expansion of $(1+x)^{n-1}$ show that **2**

$$\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\cdots+\binom{n-1}{n-2}=2^{n-1}-2$$

(ii) Find the least positive integer, n, such that **2**

$$\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\cdots+\binom{n-1}{n-2}>1000$$

(b) A block of ice in the form of a square prism is melting so that its surface area is uniformly decreasing at a rate of 10 cm^2 per second. Initially, the ice block has dimensions 30 cm, 30 cm, 120 cm. **4**

When the square side of the ice block is 25 cm, find the rate of decrease of its volume.

You may assume that as the ice melts, the dimensions of the square prism stay in the same ratio.

(c) Ten integers between 1 and 100 are randomly chosen to form the set S. Prove that there exists at least two subsets of S whose elements sum to the same value. **2**

End of Examination.

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Mathematics Extension 1

Task 1

26th November, Thursday

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Section I

1 There are $\binom{8}{3}$ ways in which you can choose the men and $\binom{6}{4}$ ways in which you can choose the men. The product of these is the number of ways that the committee can be formed.

(C)

2 The coefficient of x^3 is $\binom{5}{3} \times 2^2 = 40$

(C)

3 Choosing 13 socks is insufficient as 12 red socks and 1 blue sock could be drawn. You must draw 14 socks to ensure that you have at least 2 blue socks.

(D)

Marker's Feedback

Section II

Question 4 (8 marks)

(a) How many distinct eight letter arrangements can be made using the letters of the word PARALLEL? **1**

Sample Answer		**Marking Criteria**
(a)	$\dfrac{8!}{2!\,3!} = 3360$	1 mark

Marker's Feedback
This question was generally done well.

(b) If ${}^{2n}\mathbf{P}_{n-1} = 10 \times {}^{2n-1}\mathbf{P}_{n-2}$, find the value of n. **2**

Sample Answer		**Marking Criteria**
(b)	${}^{2n}\mathbf{P}_{n-1} = 10 \times {}^{2n-1}\mathbf{P}_{n-2}$	
	$\dfrac{(2n)!}{[2n-(n-1)]!} = \dfrac{10 \times (2n-1)!}{[2n-1-(n-2)]!}$	1 mark – expressing in factorial notation
	$\dfrac{(2n)!}{(n+1)!} = \dfrac{10 \times (2n-1)!}{(n+1)!}$	
	$\dfrac{(2n)!}{(2n-1)!} = 10$	
	$2n = 10$	1 mark – answer for n
	$\therefore n = 5$	

Marker's Feedback

This question was generally done well.

1 mark was awarded to students who correctly rewrote ${}^{2n}\mathbf{P}_{n-1} = 10 \times {}^{2n-1}\mathbf{P}_{n-2}$ in factorial notation.

(c) Ten people are attending a dinner party. The ten people will be seated at a round table.

(i) In how many ways can the ten people be arranged around the table? **1**

(ii) Two guests, Mark and Cecil wish to sit next to each other. In how many ways can the ten people be arranged such that Mark and Cecil are sitting next to each other? **1**

Sample Answer			**Marking Criteria**
(c)	(i)	$9! = 362880$	1 mark
	(ii)	$2 \times 8! = 80640$	1 mark

Marker's Feedback

This question was generally done well.

(d) The amount P of a medication in a patient, after being injected, decreases at a rate proportional to the amount present.

The initial amount of medication injected into the patient is 2 mL and the hourly rate of decrease is 3% of the drug present, at the beginning of that hourly period.

(i) How much of the drug, correct to 2 decimal places, still remains in the body after 5 hours? **1**

(ii) Find the rate of decrease of the drug when $t = 5$, correct to two significant figures. **1**

Sample Answer			**Marking Criteria**
(d)	$P = 2e^{-0.03t}$		
	(i)	$t = 5$ $P = 2e^{-0.03\times 5}$ $= 1.72$ mL	1 mark – correct answer
	(ii)	$\frac{dP}{dt} = kP = -0.03$ $= -0.03 \times 1.72 \ldots$ $= -0.052$ mL/hour	1 mark – correct answer

Marker's Feedback

This question was not done very well. Students did not **read and understand** the question. If the medication is *decreasing at a rate that is proportional to the amount present*, then we use the form $P = Ae^{-kt}$. Many students assumed that the equation $P = 2 \times 0.97^5$ gave the amount of medication present.

(e) In how many ways can you spell the word NOON in the grid below? **1**

You can start on any letter then on each step move one step in any direction (left, right, up, down, or diagonally).

You can return to the same 'N' from which you started.

N	**N**	**N**	**N**
N	**O**	**O**	**N**
N	**O**	**O**	**N**
N	**N**	**N**	**N**

Sample Answer		**Marking Criteria**
(e)	$4 \times 1 \times 3 \times 5 + 8 \times 2 \times 3 \times 5 = 300$	1 mark – correct answer and working

Marker's Feedback

This question was done poorly. Many students overestimated or underestimated the number of ways in which the word "NOON" could be formed.

Students needed to consider the two cases – where the first "N" is in the corner or on the side.

Question 5 (9 marks)

(a) Find the coefficient of x^8 in the expansion $\left(\frac{2}{x}+x^3\right)^{20}$. **2**

Sample Answer		**Marking Criteria**
(a)	$\binom{20}{r}\left(\frac{2}{x}\right)^{20-r}(x^3)^r$ Solving for r, $\left(\frac{1}{x}\right)^{20-r}(x^3)^r = x^8$ $\frac{x^{3r}}{x^{20-r}} = x^8$ $x^{3r-20+r} = x^8$ Equating powers, $4r - 20 = 8$ $4r = 28$ $r = 7$ Coefficient of x^8 $\binom{20}{7}(2)^{20-7} = \binom{20}{7}(2)^{13}$	1 mark – working towards 1 mark – finding the correct coefficient

Marker's Feedback
This question was generally done well.

(b) Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$. **2**

<table>
<tr><th colspan="2">Sample Answer</th><th>Marking Criteria</th></tr>
<tr><td>(b)</td><td>

$$\begin{aligned} LHS &= \binom{2n}{2} \\ &= \frac{(2n)!}{2!\,(2n-2)!} \\ &= \frac{2n(2n-1)}{2} \\ &= n(2n-1) \\ &= 2n^2 - n \\ &= n^2 - n + n^2 \\ &= \frac{n!}{(n-2)!} + n^2 \\ &= 2\frac{n!}{(n-2)!\,2!} + n^2 \\ &= 2\binom{n}{2} + n^2 \\ &= RHS \end{aligned}$$

Alternatively,
If we consider two groups of n and choose some combination from the two groups that result in two items from the $2n$, then we have

$$\begin{aligned} \binom{2n}{2} &= \binom{n}{0}\binom{n}{2} + \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{0} \\ &= \binom{n}{2} + n^2 + \binom{n}{2} \\ &= 2\binom{n}{2} + n^2 \end{aligned}$$

</td><td>2 marks – correctly showing that the LHS is equal to the RHS

1 mark – adequate working towards full solution</td></tr>
<tr><td colspan="3">Marker's Feedback
This question was done poorly. Many students made leaps in logic that did not make sense when using factorial notation. No students used the alternate method of working out.

1 mark was allocated if students were able to correctly write $\binom{2n}{2}$ in factorial notation.</td></tr>
</table>

(c) A girls' volleyball team has 14 players, including a set of triplets: Amanda, Amy and Anna. **1**

In how many ways can we choose 6 starters if at least one of the triplets must be in the starting line-up?

Sample Answer		**Marking Criteria**
(c)	$\binom{14}{6} - \binom{11}{6} = 2541$	1 mark – correct answer and working

Marker's Feedback

This question was a straight forward question that should have been done better.

(d) During summer, the population of fruit flies, P, is observed. The rate of increase of the population per week is modelled by the equation

$$\frac{dP}{dt} = k(P - 3000)$$

where k is a constant. At the beginning of summer, the population is 4000 and after 4 weeks, it is 10 000.

(i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, where A is a constant and t is the time in weeks. **1**

(ii) Find the values of A and k. **2**

(iii) After how many weeks does the population reach half a million? Give your answer correct to 1 decimal place. **1**

Sample Answer			**Marking Criteria**
(d)	$t = 0, P = 4000$		
	(i)	$P = 3000 + Ae^{kt}$ Now, $\frac{dP}{dt} = kAe^{kt}$ Since, $Ae^{kt} = P - 3000$ $\therefore \frac{dP}{dt} = k(P - 3000)$	1 mark –showing correct steps
	(ii)	$4000 = 3000 + Ae^{0}$	1 mark – solving for A

		$\therefore A = 1000$ $P = 3000 + 1000e^{kt}$ When $t = 4, P = 10\ 000$ $10000 = 3000 + 1000e^{4k}$ $7000 = 1000e^{4k}$ $7 = e^{4k}$ $\ln(7) = 4k$ $k = \dfrac{\ln(7)}{4}$	1 mark – solving for k and leaving answer in exact form
	(iii)	$P = 500\ 000$ $500000 = 3000 + 1000e^{0.486\ldots t}$ $497000 = 1000e^{0.486\ldots t}$ $\ln(497) = 0.486\ldots t$ $t = 12.76\ldots$ $t \approx 12.8$ weeks	1 mark – correct working and final answer

Marker's Feedback

This was generally well done.

In (i) make sure you are showing the substitution CLEARLY, either by making Ae^{kt} the subject of the initial equation, or by adding and subtracting a 3000 to make the initial equation extremely visible.

In part (ii) you should NEVER be giving a decimal approximation unless asked in the question. The mark was awarded if you demonstrated that you were using the exact representation in the next question, however, there should still not be an approximation.

Question 6 (10 marks)

(a) The radius of a circular spill (r km), at a time t hours after it was first observed is given by $r = \dfrac{1 + 3t}{1 + t}$. Find the exact rate of increase of the area of the oil spill when the radius is 2 kilometres. **3**

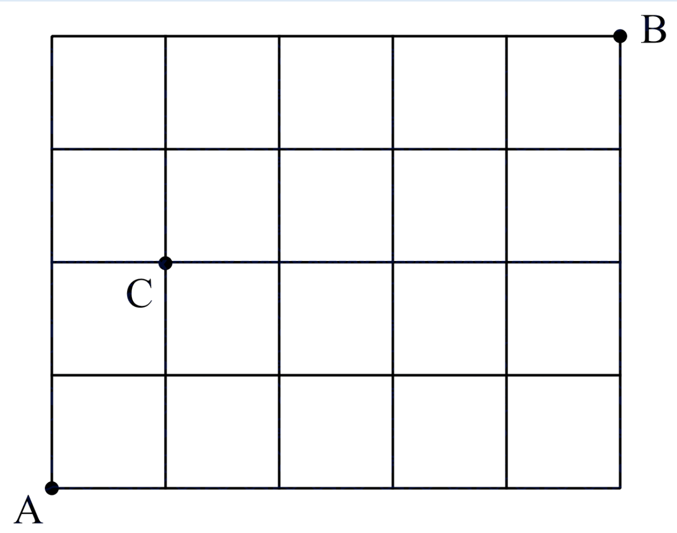
Sample Answer		Marking Criteria
(a)	$r = \dfrac{1+3t}{1+t}$ $u = 1 + 3t \qquad v = 1 + t$ $u' = 3 \qquad v' = 1$ $\dfrac{dr}{dt} = \dfrac{3(1+t) - (1-3t)}{(1+t)^2}$ $= \dfrac{3 + 3t - 1 - 3t}{(1+t)^2}$ $= \dfrac{2}{(1+t)^2}$	1 mark – correctly differentiating
	When $r = 2$ $2 = \dfrac{1+3t}{1+t}$ $2 + 2t = 1 + 3t$ $t = 1$ So, $\dfrac{dr}{dt} = \dfrac{2}{(1+1)^2}$ $= \dfrac{1}{2}$	1 mark – finding t and finding the value of $\frac{dr}{dt}$
	$\dfrac{dA}{dt} = \dfrac{dA}{dr} \times \dfrac{dr}{dt}$ $= 2\pi r \times \dfrac{1}{2}, \qquad r = 2$ $= 2\pi \text{ km}^2/h$	1 mark – using chain rule and substituting relevant values to find $\frac{dA}{dt}$

Marker's Feedback

While most attempts showed a clear understanding of the process, many students failed to find a value for t, when r=2.

Students who attempted to eliminate t, by substituting the initial equation generally made small errors due to the increased complexity of the calculation they were now making.

(b) A nine-step path is drawn from A to B in the diagram below. If it is drawn randomly what is the probability it passes through C? **2**



Sample Answer		Marking Criteria
(b)	$$\frac{\binom{3}{2}\times\binom{6}{2}}{\binom{9}{4}}=\frac{45}{126}$$	1 mark – number paths to C or total possible paths 1 mark – writing as a probability
Marker's Feedback Very few students attempted this question. A mark was awarded for breaking the question in to "rights" and "ups" or equivalent.		

(c) The motion of a particle satisfies the displacement-time equation $x = 4t^2 e^{-t}$. **5**

Describe its motion with reference to:

- When the particle changes direction and where
- The displacement as $t \to \infty$
- The velocity as $t \to \infty$
- Whether the particle speeds up or slows down as $t \to \infty$

Sample Answer		**Marking Criteria**
(c)	$x = 4te^{-2t}$ $u = 4t \qquad v = e^{-2t}$ $u' = 4 \qquad v' = -2e^{-2t}$	
	$v = 4e^{-2t} - 8te^{-2t}$ $= 4e^{-2t}(1 - 2t)$ $a = -8e^{-2t} - 2(4e^{-2t} - 8te^{-2t})$ $= -16e^{-2t} + 16te^{-2t}$ $= -16\mathrm{e}^{-2t}(1 - \mathrm{t})$	1 mark – calculating derivatives
	Particle changes direction $v = 0$, $4e^{-2t}(1 - 2t) = 0$ $4e^{-2t} = 0, \qquad 1 - 2t = 0$ $t = \frac{1}{2}$ and there are no solutions for $4e^{-2t} = 0$ Maximum speed reached when $a = 0$ $-16e^{-2t}(1 - t) = 0$ $-16e^{-2t} = 0, \qquad 1 - t = 0$ $t = 1$ and there are no solutions for $-16e^{-2t} = 0$ When $t = 0, x = 0$ As $t \to \infty$, $x \to 0^+$ $v \to 0^- \Rightarrow v > 0$ $a \to 0^+ \Rightarrow a < 0$	1 mark – solving $v = 0$ and $a = 0$
	<u>Initially the particle is at the origin and it moves away in the positive direction.</u> The particle is slowing down on $0 < t < \frac{1}{2}$ and <u>changes direction at $t = \frac{1}{2}$</u>. It then speeds up heading toward the negative direction. <u>The particle reaches its max speed at $t = 1$, after which it starts slowing down as $t \to \infty$.</u>	3 marks – one mark each for the underlined components

Marker's Feedback

Too many students forgot to use the product rule. Without the product rule the explanations become too simple and only mark that could be awarded was for the explanation of the behaviour

of x as it heads towards infinity. Students also needed to recognise that a description of the motion should include the initial conditions.

Question 7 (10 marks)

(a) (i) Using the expansion of $(1+x)^{n-1}$ show that **2**

$$\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\cdots+\binom{n-1}{n-2}=2^{n-1}-2$$

(ii) Find the least positive integer, n, such that **2**

$$\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\cdots+\binom{n-1}{n-2}>1000$$

Sample Answer			**Marking Criteria**
(a)	(i)	$(1+x)^{n-1}=\binom{n-1}{0}1^{n-1}x^0+\binom{n-1}{1}1^{n-2}x+\cdots+\binom{n-1}{n-2}1x^{n-2}$ $+\binom{n-1}{n-1}1x^{n-1}$ Let $x=1$, so $(1+1)^{n-1}=\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{n-2}+\binom{n-1}{n-1}$ $2^{n-1}=1+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-3}+\binom{n-1}{n-2}+1$ $2^{n-1}-2=\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-3}+\binom{n-1}{n-2}$	1 mark – writing the expansion 1 mark – substituting $x=1$ and rearranging to acquire required equation
	(ii)	$\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-3}+\binom{n-1}{n-2}>1000$ Then, from (i) $2^{n-1}-2>1000$ $2^{n-1}>1002$ $\ln(2)^{n-1}>\ln(1002)$ $n-1>\dfrac{\ln(1002)}{\ln(2)}$ $n>\dfrac{\ln(1002)}{\ln(2)}+1$ $n>10.968\ldots$ $n\approx 11$	1 mark – using appropriate substitution for series 1 mark – applying log laws correctly 1 mark – correct rounded answer

Marker's Feedback
Part (i) - generally insufficient working shown. Part (ii) - Many students rushed and found $\ln 998$ instead of $\ln 1002$. Generally insufficient working shown. Students must reference from part (i). Students must explain how they can calculate $\log_2 1002$.

(b) A block of ice in the form of a square prism is melting so that its surface area is uniformly decreasing at a rate of 10 cm^2 per second. Initially, the ice block has dimensions 30 cm, 30 cm, 120 cm. **4**

When the square side of the ice block is 25 cm, find the rate of decrease of its volume.

You may assume that as the ice melts, the dimensions of the square prism stay in the same ratio.

Sample Answer		**Marking Criteria**
(b)	Suppose the dimensions of the square prism are x, x and $4x$, then, $A = 2x^2 + 4(4x^2)$ $= 2x^2 + 16x^2$ $= 18x^2$	1 mark – finding expression for A and V
	$\frac{dA}{dx} = 36x$ $V = x^2 \times 4x$ $= 4x^3$ $\frac{dV}{dx} = 12x^2$	1 mark –differentiating A and V with respect to x
	$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$ $= 12x^2 \times \frac{1}{36x} \times (-10)$ $= -\frac{10x}{3}$	1 mark – using chain rule and substituting to find expression for $\frac{dV}{dt}$
	When $c = 25$, $\frac{dV}{dt} = -\frac{250}{3}$ cm^3/sec	1 mark – making substitution to find value of $\frac{dV}{dt}$
Marker's Feedback		

Question very poorly done. If students got to this point, most did not approach the question with a plan.

(c) Ten distinct integers between 1 and 100 are randomly chosen to form the set S. Prove that there exists at least two subsets of S whose elements sum to the same value. **2**

	Sample Answer	Marking Criteria
(c)	Sum of smallest subset possible is 2 Sum of largest subset possible $S_1 = \{90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$ $\|S_1\| = 945$ Number of possible subsets is $2^{10} - 1 = 102$, as empty set must be excluded, i.e. $\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \cdots + \binom{10}{9} + \binom{10}{10}$ since $945 < 1023$, by the Pigeonhole Principle there must exist at least 2 subsets whose elements sum to the same value.	1 mark – total number of subsets or calculating the sum of the smallest AND largest possible subsets. 1 mark – for comparison between pigeons and pigeonholes

Marker's Feedback This question was rarely done but when attempted, demonstrations were poor and often didn't reference the pigeonhole principle or referred to it as PHP.