Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2021 **YEAR 11**

Extension Mathematics

Task 1

23rd November 2021

General Instructions:

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using blue or black pen
- Only NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: **40**

Q	Marks
1	12
2	16
3	12
Total	40

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

BLANK PAGE

Examination continues on next page

Section II

In Questions 1 – 3, your response should include relevant mathematical reasoning and/or calculations.

Question 1 (12 marks) Answer each question in a NEW booklet.

(a) Steve is putting 12 cups in a row. Given that there are 4 identical red cups, 4 identical blue cups and 4 identical green cups. How many possible arrangements of cups are there? **1**

(b) Prove that $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$ **2**

(c) Find the coefficient of x^3 in $(1+2x)^{14}$ **2**

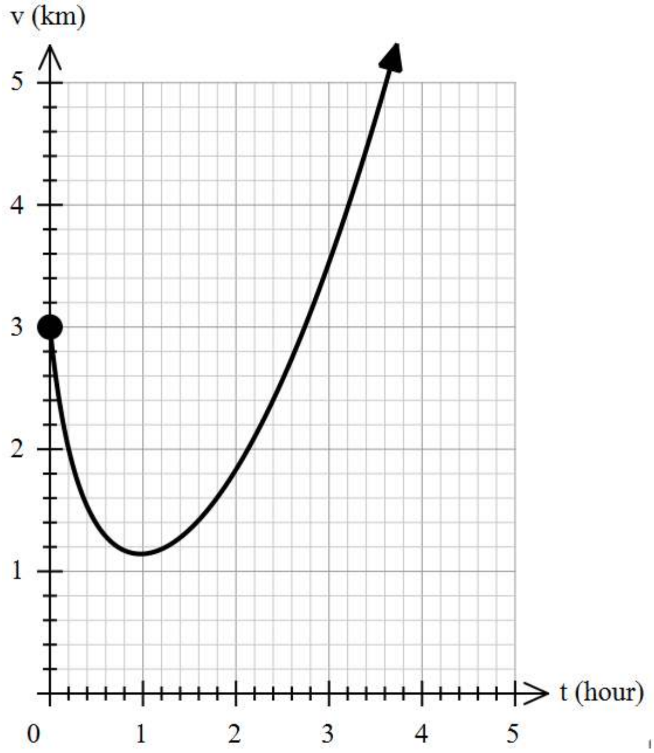
(d) A set of blocks contains red, blue, yellow and green blocks. How many blocks must be chosen to ensure that there are at least 5 blocks that are the same colour? **1**

(e) Prove by Mathematical Induction for all integers $n \geq 1$. **3**

$$5^n + 2 \times 11^n \text{ is a multiple of 3.}$$

Examination continues on next page

(f) A car travels from Woy Woy to Sydney, the velocity graph represents a part of the journey.



(i) Find the initial velocity. **1**

(ii) Describe the acceleration of the car from the above velocity graph. **2**

Examination continues on next page

Question 2 (16 marks) Answer each question in a NEW booklet.

(a) 8 boxes are arranged in a row. In how many ways can 5 distinct balls be placed into them, given that no box can hold more than one ball? **1**

(b) A club has 25 members, and wishes to pick a President, Secretary, and Treasurer. In how many ways can we choose President, Secretary, and Treasurer, if:

(i) Individual members can only hold at most 1 position? **1**

(ii) Individual members may hold all 3 positions? **1**

(iii) Individual members can hold at most 2 positions? **1**

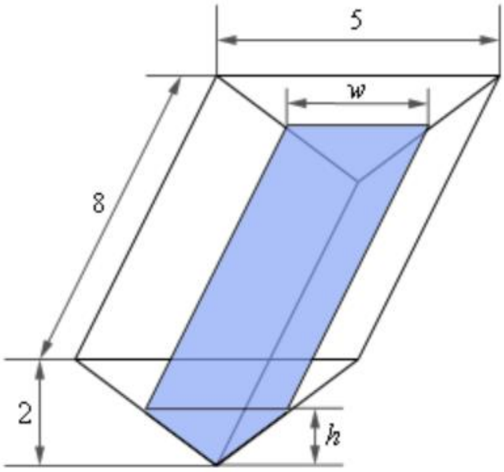
(c) 9 people are sitting around a circular table.

(i) How many possible ways are there for them to sit around the table? **1**

(ii) What is the probability that Jake, Amy and Rosa all sit together? **2**

Examination continues on next page

(d) A water trough is 8 meters in length and its ends are in the shape of isosceles triangles with a width of 5 metres and a perpendicular height of 2 metres. If water is being pumped in at a constant rate of $6\text{m}^3/\text{sec}$:



(i) Explain why $5h = 2w$ **1**

(ii) Find the rate at which the water level is rising at the moment when the water has a height of 120 cm. **2**

(iii) Find the rate of change of the width when the water has a height of 120cm. **1**

(e) A freshly caught fish, initially at 16°C, is placed in a freezer that has a constant unknown temperature of m°C. The cooling rate of the fish is proportional to the difference between the temperature of the freezer and the temperature T° C of the fish. It is known that T satisfies the equation $\frac{dT}{dt} = -k(T - m)$, where t is the number of minutes after the fish is placed in the freezer.

(i) Show that $T = m + Ae^{-kt}$ satisfies the equation. **1**

(ii) If the temperature of the fish is 8°C after 5 minutes, find the value of k. In terms of m. **3**

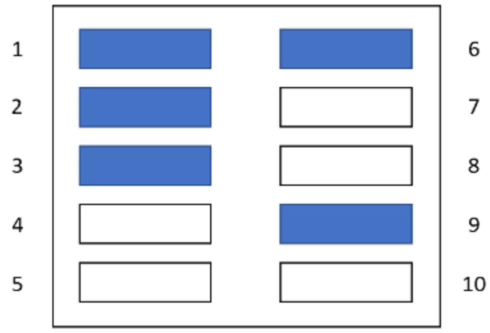
Examination continues on next page

(iii) Find the temperature of the fish after half an hour when the initial temperature of the freezer is $-2°C$. **1**

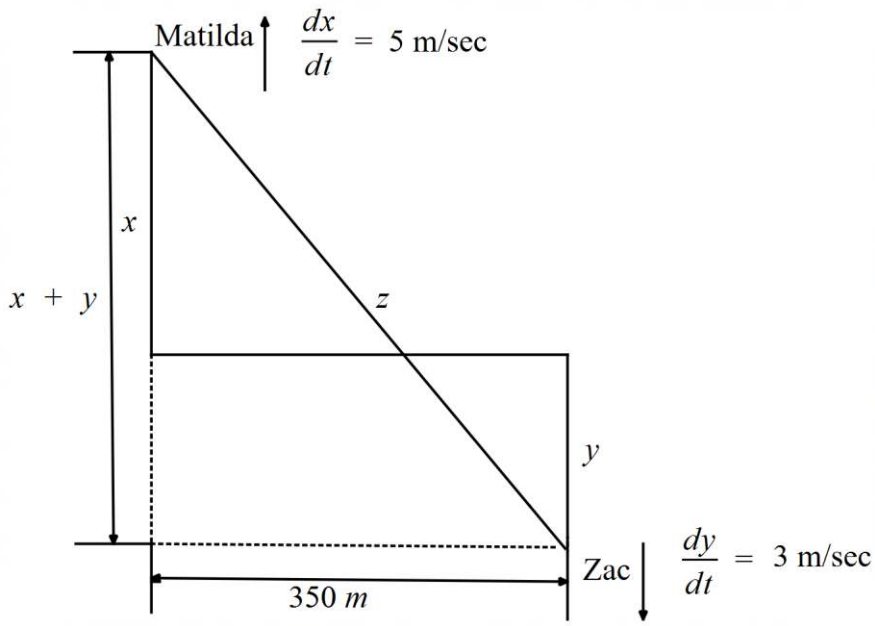
Examination continues on next page

Question 3 (12 marks) Answer each question in a NEW booklet.

(a) A commercially available ten-button lock may be opened by depressing (in any order) the correct distinct 5 buttons. Once a button is depressed, or pushed in, it will stay pushed in. The sample shown has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose these locks are redesigned so that sets of as many as 9 buttons, or as few as 1 button could serve as combinations. How many additional combinations would this allow? **2**



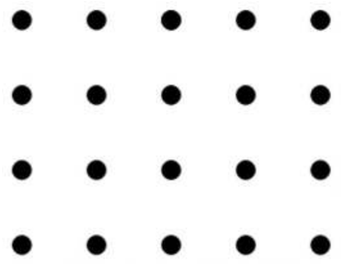
(b) Zac is 350m directly east of Matilda. Matilda starts riding her bike north at a rate of 5 m/sec and 7 minutes later Zac starts riding his bike south at 3 m/sec. At what rate is the distance separating the two people changing 25 minutes after Zac starts riding? Answer to nearest m/s. **3**



Examination continues on next page

(c) Show that among any 51 points inside a unit square, there will always be at least three points that can be covered by a circle of radius $\frac{1}{7}$. **2**

(d) Four points are randomly chosen from the grid pictured to the right. What is the probability that they are the vertices of a rectangle whose sides are parallel to the sides of the grid? **2**



(e) Prove the following binomial theorem by Mathematical Induction **3**

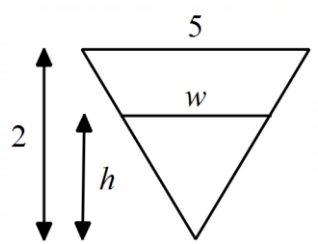
$$(a + b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r, \text{ n} \geq 1.$$

End of Examination

Examination continues on next page

Question 1	Marking Criteria	Marking Feedback
a) $\frac{12!}{(4!)^3} = 34650$		Majority of students got correct answer.
b) $LHS = \binom{n}{r}$ $= \frac{n!}{(n-r)!\,r!}$ $= \frac{n(n-1)!}{(n-r)!\,r(r-1)!}$ $= \frac{n}{r} \times \frac{(n-1)!}{\big(n-1-(r-1)\big)!\,(r-1)!}$ $= \frac{n}{r}\binom{n-1}{r-1}$ $= RHS$	1 mark for correct use of factorial notation to represent combinations 2 marks correctly proves using only one side	Some of students tend to prove a halfway from LHS and a halfway from RHS, instead of prove all the way from either LHS to RHS or from RHS to LHS.
c) $\binom{14}{3}(1)^{11}(2)^3 = 364 \times 8$ $= 2912$	1 mark attempts to use the binomial theorem to expand. 2 marks correctly uses the binomial expansion for both the coefficient of x and the combination.	In general, done well. With a few students by substituting x = 1, instead of recognising the coefficient of x^3.
d) If there are 17 blocks (pigeons) and 4 colours (pigeonholes) there must be at least $\frac{17}{4} = 4.25$ (i.e. 5) of the same colour.	Correctly identifies 17 blocks, with appropriate justification,	Quite a few students got 21 blocks instead of 17 blocks. Many students did not provide any justification or reasoning.
e) When $n = 1$ $LHS = 5^1 + 2 \times 11^1$ $= 5 + 22$ $= 27$ Which is divisible by 3 Assume true for $n = k$ $5^k + 2 \times 11^k = 3M$ for $M \epsilon \mathbb{N}$ Then $5^k = 3M - 2 \times 11^k$ Consider $n = k + 1$ $LHS = 5^{k+1} + 2 \times 11^{k+1}$ $= 5 \times 5^k + 22 \times 11^k$ $= 5\big(3M - 2 \times 11^k\big) + 22 \times 11^k$ by assumption	*Physical working for $n = 1$ *Concluding statement for $n = 1$. “true for n=1” *Assumption for $n = k$ **Algebra and physical working for $n = k + 1$ (must include “by assumption” or equivalent) *Conclusion, **including $\boldsymbol{n = k}$ and $\boldsymbol{n = k + 1}$** conclusion and that $n \geq 1$ All 6 *’s – 3 marks At least 4 *’s – 2 marks	Done poorly in general. 1 mark is deducted if any of the required statement is missing.

$= 5 \times 3M - 10 \times 11^k + 22 \times 11^k$ $= 5 \times 3M + 12 \times 11^k$ $= 3(5M + 4 \times 11^k)$ $\therefore n = k$ implies $n = k + 1$ $\therefore 5^n + 2 \times 11^n$ is divisible by 3 for all $n \epsilon \mathbb{N}$ by the principles of mathematical induction.	At least 2 *'s – 1 mark Leniency shown for including n=k implies n=k+1	
f) $v(0) = 3\text{km/h}$		Done well in general.
Ii) Initially the acceleration is against the motion of the car (left/negative), the magnitude of the acceleration decreases until there is no force applied to the car at $t = 1$. The car then accelerates in the direction of motion (right/positive) at an increasing rate.	1 mark – Accurately describes at least 1 portion of the graph 2 marks – Accurately describes all sections of the graph	Quite a few students were just reading from the velocity graph, instead of describing acceleration from the velocity graph.

Question 2	Marking Criteria	Marking Feedback
(a) $^{8}C_5 \times 5! = 6720$	1 mark correct answer from correct working	Students didn't seem to understand the difference between $^{8}C_5$ and $^{8}P_5$. Permutations consider the order of the items being arranged. Combinations consider the grouping of the items.
(b) (i) $^{25}P_3 = 13800$	1 mark correct answer from correct working	Done well.
(ii) $25^3 = 15625$	1 mark correct answer from correct working	Done well.
(iii) $25^3 - {}^{25}C_1 = 15600$	Can carry forward error from part (ii)	Not done well.
(c) (i) $8! = 40320$	1 mark correct answer from correct working	Done well.
(ii) $\text{Total sitting together} = 3! \times 6!$ $\text{P(together)} = \dfrac{3! \times 6!}{8!}$ $= \dfrac{3}{28}$	1 mark – either correctly calculated the arrangements where 3 are together, or made a small error in arrangements 2 marks – probability	Many students did not calculate a **probability** and simply calculated the number of arrangements.
(d) (i)  By similar triangles, $\dfrac{h}{2} = \dfrac{w}{5}$ $5h = 2w$	1 mark refers to similar triangles	In general, if students mentioned the **sides** were in ratio or in proportion, then they were awarded the mark.

(ii)	Find $\frac{dh}{dt}$ when $h = 120$ cm $= 1.2$ m $$\frac{dV}{dt} = 6$$ $$V = \frac{1}{2}h \times w \times 8$$ $$= 4 \times h \times \left(\frac{5h}{2}\right)$$ $$= 10h^2$$ $$\frac{dV}{dh} = 20h$$ $$\frac{dh}{dV} = \frac{1}{20h}$$ $$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$ $$= \frac{6}{20h}$$ When $h = 1.2$ m $$\frac{dh}{dt} = \frac{6}{20 \times 1.2}$$ $$= \frac{1}{4} \text{ m/sec}$$	1 mark – for finding V in terms of h. 2 marks – for correctly finding V, $\frac{dV}{dh}$ and multiplying this rate by $\frac{dV}{dt}$	Not done well. Many students didn't realise h and w vary as the water level increases and substituted a value in for h before differentiating which meant they did not calculate a correct rate. 1 mark deducted for substituting $h = 120$ instead of $h = 1.2$
(iii)	$$\frac{dh}{dt} = \frac{6}{20h}$$ We know $w = \frac{5h}{2}$ So $\frac{dw}{dh} = \frac{5}{2}$ $$\frac{dw}{dt} = \frac{dw}{dh} \times \frac{dh}{dt}$$ $$= \frac{5}{2} \times \frac{6}{20h}$$ $$= \frac{6}{8h}$$ When $h = 1.2$	1 mark for correct answer from correct working.	Not done well. Students in general demonstrated a poor understanding of related rates.

$= \frac{5}{8}$ m/sec		
(e) (i) $T = m + Ae^{(-kt)}$ and $Ae^{-kt} = T - m$ so $\frac{dT}{dt} = -k\left(Ae^{-kt}\right)$ $= -k(T - m)$	1 mark for correct working leading to required result.	Mostly done well, however, there were a handful of students with poor setting out. Students must pay attention to how their working logically flows.
(ii) $T(0) = m + A = 16$ $T = m + A = 16$ $A = 16 - m$ $T(5) = m + (16 - m)e^{-5k} = 8$ $e^{-5k} = \frac{8 - m}{16 - m}$ $-5k = \ln\left(\frac{8 - m}{16 - m}\right)$ $k = \frac{1}{5}\ln\left(\frac{16 - m}{8 - m}\right)$	3 marks for correctly finding A and then applying log laws to find the value of k in terms of m. 1 mark deducted for not finding A correctly. 1 mark for any errors in working	This was mostly done well.
(iii) Let $m = -2$ $k = \frac{1}{5}\ln\left(\frac{18}{10}\right)$ $= \frac{1}{5}\ln\left(\frac{9}{5}\right)$ $T(0) = -2 + A = 16$ $A = 18$ $T(30) = -2 + 18e^{-30k}$ $= -1.4707 \ldots$	1 mark for correct answer from correct working.	Mostly done well if (ii) was done well.

Question 3	Marking Criteria	Marking Feedback
a) $\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\cdots+\binom{10}{9}-\binom{10}{5}=2^{10}-2-\binom{10}{5}$ $=770$	1 mark – identifies more combinations 2 marks – finds answer	Many students assumed that because it was a lock it must be a permutation. Is a combination. Many students correctly calculated the amount of possible combinations but did not subtract the original combinations in order to get the NEW combinations.
b) let $x+y=w$ Consider $t=0$ to be when person A starts riding $\frac{dw}{dt}=8$ $350^2+w^2=z^2$ $z=\sqrt{350^2+w^2}$ $\frac{dz}{dw}=\frac{2w}{2}(350^2+w^2)^{-\frac{1}{2}}$ $\frac{dz}{dt}=\frac{dz}{dw}\times\frac{dw}{dt}$ $=8w(350^2+w^2)^{-\frac{1}{2}}$ $After\ 25\ minutes\ t=1500$ $w=8(1500)+7\times 60\times 5$ $=14100$ sub into $\frac{dz}{dt}$ $\frac{dz}{dt}=8\times 1500\times(350^2+(1500)^2)^{-\frac{1}{2}}$ $=12000(2372500)^{-\frac{1}{2}}$ $\approx 8\text{ms}^{-1}$	1 mark – find an expression for z in terms of $x+y$ 2- Finds an expression for $\frac{dz}{dt}$ 3- Final answer	Poorly done. Many students used minutes, did not convert to seconds. Many students found z in terms of x + y, but did not follow through to find the rate of change of z in terms of x+y. Instead finding the value in terms of x or y.
c) Divide the square into 25 equal squares. Each will have a side length of $\frac{1}{25}$ and a diagonal of $\frac{\sqrt{2}}{25}$ which is less that $\frac{2}{7}$. Therefore, each square would be completely contained within a circle of radius $\frac{1}{7}$. Given that we have 25 smaller squares (pigeonholes) and 51 points (pigeons). At least one smaller square will have three points in it, and all three can then be contained within a circle of radius $\frac{1}{7}$.		Many students, seeing the 7, divided into a 7x7, and they could not appropriately justify why a circle would then have to cover 3 points.

d) Choose 2 rows from four and two columns from 5. This will form your rectangle. Choose 4 points from 20 for your random points. $P(Rect) = \frac{{}^{4}C_2 \times {}^{5}C_2}{{}^{20}C_4}$ $= \frac{4}{323}$	1 for some attempt at probability with the correct denominator. 2 for correct answer	Many methods were used to find the total number of rectangles. Most students correctly identified the denominator of the probability (prior to simplification).
e) Show true for $n=1$ $LHS = (a+b)$ $RHS = \sum_{r=0}^{1} \binom{1}{r} a^r b^{1-r}$ $= \binom{1}{0} a^0 b^1 + \binom{1}{1} a^1 b^0$ $= a+b = RHS$ $\therefore$ The statement is true for $n=1$ Assume true for $n=k$ $(a+b)^k = \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r}$ $= \binom{k}{0} b^k + \binom{k}{1} a b^{k-1} + \binom{k}{2} a^2 b^{k-2} + \ldots + \binom{k}{k} a^k$ Prove true for $n=k+1$ $LHS = (a+b)^{k+1}$ $= (a+b)(a+b)^k$ $= (a+b)\left(\binom{k}{0} b^k + \binom{k}{1} a b^{k-1} + \binom{k}{2} a^2 b^{k-2} + \binom{k}{k} a^k\right)$ (by assumption) $= \binom{k}{0} a b^k + \binom{k}{1} a^2 b^{k-1} + \binom{k}{2} a^3 b^{k-2} + \ldots \binom{k}{k} a^{k+1} +$ $\binom{k}{0} b^{k+1} + \binom{k}{1} a b^k + \binom{k}{2} a^2 b^{k-1} + \ldots + \binom{k}{k} b a^k$	*Physical working for $n = 1$ *Concluding statement for $n = 1$. "true for n=1" *Assumption for $n = k$ **Algebra and physical working for $n = k + 1$ (must include "by assumption" or equivalent) *Conclusion, **including $n = k$ and $n = k + 1$** conclusion and that $n \geq 1$ All 6 *'s – 3 marks At least 4 *'s – 2 marks At least 2 *'s – 1 mark Leniency shown for including n=k implies n=k+1	Many students did not attempt this question Of those that attempted the question many students put the shape of an induction question around working that was just using the theorem we were trying to prove, or appeared to just jump to the correct final step with insufficient justification.

$= \binom{k}{0} b^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right] a b^{k} + \left[\binom{k}{1} + \binom{k}{2}\right] a^{2} b^{k-1} + \dots + \left[\binom{k}{k-1} + \binom{k}{k}\right] a^{k} b + \binom{k}{k} a^{k+1}$ As, $\binom{k}{0} = \binom{k+1}{0} = 1$ and $\binom{k}{n-1} + \binom{k}{n} = \binom{k+1}{n}$ and $\binom{k}{k} = \binom{k+1}{k+1} = 1$ $= \binom{k+1}{0} b^{k+1} + \binom{k+1}{1} a b^{k} + \binom{k+1}{2} a^{2} b^{k-1} + \dots + \binom{k+1}{k} a^{k} b + \binom{k+1}{k+1} a^{k+1}$ $= \sum_{r=0}^{k+1} \binom{k+1}{r} a^{r} b^{k+1-r}$ $=$ RHS $\therefore$ $n=k$ implies $n=k+1$ $\therefore$ By the principle of mathematical induction the statement is true for all $n \geq 1$		