Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2022 **YEAR 12**

Extension Mathematics

Task 3

16th June 2022

General Instructions:

- Reading time – 5 minutes
- Working time – 55 minutes
- Write using blue or black pen
- Only NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 38

Section I - 4 marks

- Allow about 5 minutes for this section

Section II - 34 marks

- Allow about 50 minutes for this section

Q	Marks
MC	/4
5	/13
6	/10
7	/11
Total	/38

This question paper must not be removed from the examination room.

This assessment task constitutes 30% of the course.

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Section I

4 marks
Allow about 5 minutes for this section

Use the multiple-choice sheet for Question 1–4

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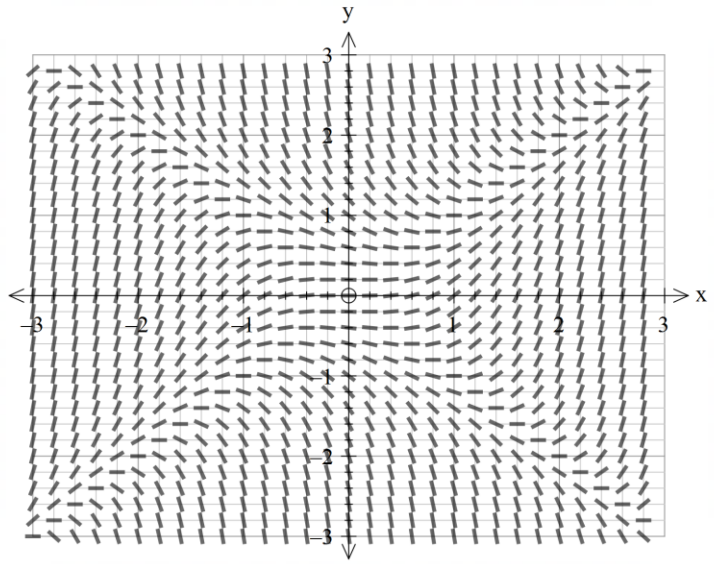
1 The graph below is the direction field for a first order differential equation. The differential equation could be:



(A) $y' = y - x$

(B) $y' = x^2 - y^2$

(C) $y' = y^2 - x^2$

(D) $y' = x^2 + y^2$

Examination continues on next page

2 Using the substitution $u = \sqrt{x+1}$ then the integral below can be expressed as

$$\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}}$$

(A) $$\int_1^{\sqrt{3}} \frac{1}{\sqrt{u}(u^2+1)}\,du$$

(B) $$\int_0^2 \frac{2}{u^2+1}\,du$$

(C) $$\int_1^3 \frac{1}{\sqrt{u}(u+1)}\,du$$

(D) $$2\int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$$

3 Which of the following is a first order separable equation

(A) $$\frac{dy}{dx} = \frac{y}{x+y}$$

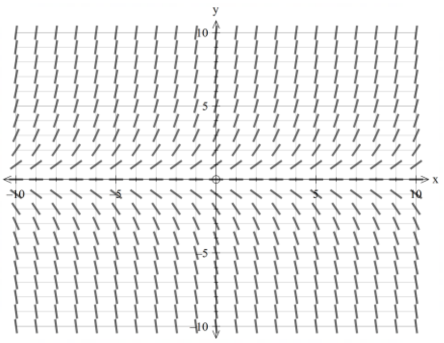
(B) $$\frac{dy}{dx} = \log_{\mathrm{e}}(x+y)$$

(C) $$\frac{dy}{dx} = e^{x+y}$$

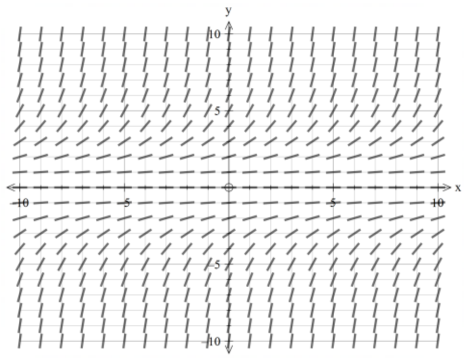
(D) $$\frac{d^2y}{dx^2} = xy$$

4 Which of the following slope fields does not represent a differential equation of the form $\frac{dy}{dx} = f(y)$?

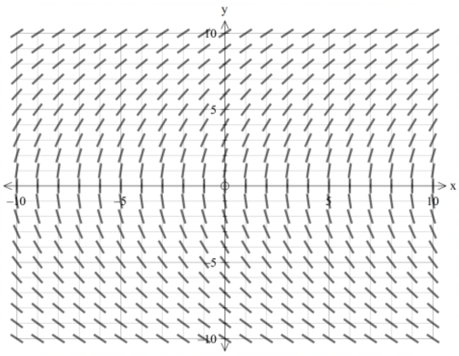
(A)



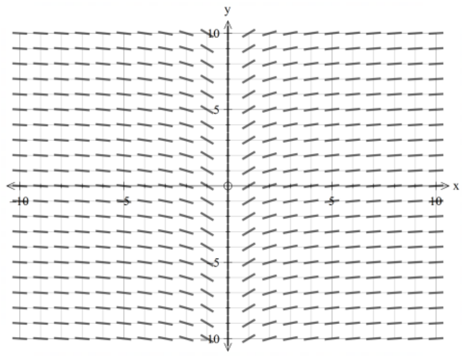
(B)



(C)



(D)



Section II

In Questions 5 – 7, your response should include relevant mathematical reasoning and/or calculations.

Question 5 (13 marks) Answer each question in a NEW booklet.

(a) In the domain $0 \leq x \leq 2\pi$, solve $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ **3**

(b) Let **2**

$$f(x) = \frac{1}{\arcsin(x)}$$

Find $f'(x)$ and state the largest set of values of x for which $f'(x)$ is defined.

(c) Find **2**

$$\int \cos^2 x\, dx$$

(d) Given that $y = e^{kt}$ is a solution to $y'' + y' - 2y = 0$, find the values of k. **3**

(e) Evaluate using the substitution $u = \cos x$ **3**

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^4 x\, dx$$

(f) Find the particular solution to the differential equation $\frac{dy}{dx} = \sec y$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ **2**

Examination continues on next page

Question 6 (10 marks) Answer each question in a NEW booklet.

(a) Find **3**

$$\int \frac{6+x}{4+x^2}\,dx$$

(b) Find the general solution to $\frac{dy}{dx} - xy = 0$ **2**

(c) Use t = tan $\frac{x}{2}$ to solve $5\sin x + 4\cos x = 5$ for $0 \leq x \leq 2\pi$. **2**

(d) When molten glass is heated to high temperatures, it can be blown to create different glass formations.
A glass blower creates a vase whose volume can be modelled by rotating the curve $y = x^2 - 50$ about the y-axis between $y = 0$ and $y = 60$.

(i) Determine the exact volume of the vase. **2**

(ii) The glassblower wants to use the vase to hold a plant that requires exactly 2800π cm^3 of soil. Solve for h, the height (in cm), to which the glassblower needs to fill the pot with soil **1**

Examination continues on next page

Question 7 (11 marks) Answer each question in a NEW booklet.
2

(a) The rate at which a poulation grows in dependent upon both the size of the population and the difference between the population and the maximum possible population that can be sustained (carrying capacity). The rate of growth of a particular popultion can be modelled by $\frac{dP}{dt} = 0.001P(100 - P)$.

(i) Show that **1**

$$\frac{1}{100P} + \frac{1}{100(100 - P)} = \frac{1}{P(100 - P)}$$

(ii) Hence, or otherwise find the general solution to $\frac{dP}{dt} = 0.001P(100 - P)$ **2**

(b) Find the smallest positive value of x, in radians, such that **2**

$$\tan(4x) = \frac{\cos x - \sin x}{\cos x + \sin x}$$

(c) Solve the differential equation $(\sin x)y' + (\cos x)\,y = \tan x$ **2**

Examination continues on next page

(d) A space probe is launched vertically upwards from the surface of a spherical planet with a radius R. If the atmospheric drag is ignored, the upwards velocity (v ms^{-1}) of the probe at height h metres above the surface of the planet is modelled by the solution of the differential equation $\frac{\text{dv}}{\text{dh}} = -\frac{gR^2}{v(R+h)^2}$, where g is the gravitational acceleration on the surface of the planet. It is also known that when $h = 0$, $v = u$.

(i) Show that **3**

$$v^2 = u^2 - \frac{2gR}{1 + \frac{R}{h}}$$

(ii) In order to escape a planet's atmosphere v^2 must remain strictly positive as the space probe ascends. Find the minimum initial velocity such that the space probe will escape earth's orbit. **1**

End of Examination

Examination continues on next page

1	B	
2	D (BOUNDS)	
3	C	
4	D	

Question 5

<table>
<tr><td>a)</td><td>For $\sin(x-\alpha)$
$r^2 = 3^2 + \left(\sqrt{3}\right)^2$
$r = \sqrt{12}$
$r = 2\sqrt{3}$
$\tan\alpha = \frac{\sqrt{3}}{3}$
$\alpha = \frac{\pi}{6}$
$2\sqrt{3}\sin\left(x-\frac{\pi}{6}\right) = \sqrt{3}$
$\sin\left(x-\frac{\pi}{6}\right) = \frac{1}{2}$
$x-\frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$
$x = \frac{\pi}{3}, \pi$

Or equivalent for $r\cos(x+\alpha)$, using t-formula or just manipulating the expression to form a compound angle.</td><td>1 mark – have found one solution with limited working have found the constants but have not found the correct solutions

2 marks – Error in finding one of the constant but otherwise correct. Or have found the two solutions but have not shown the derivation of the two constants. Or have only found 1 solution.

3 marks – have both solutions and have shown the derivation of the two constants</td></tr>
<tr><td>(b)</td><td>$f'(x) = \frac{1}{\arcsin(x)\sqrt{1-x^2}}$
Domain of $f'(x), x\epsilon(-1,1)$</td><td>One mark each item</td></tr>
<tr><td colspan="3">Generally the differentiation was done well, however, many students did not consider the division by zero in their domain</td></tr>
<tr><td>(c)</td><td>$\int \cos^2 x\, dx$
$\int \frac{1}{2}\cos 2x + \frac{1}{2} dx$
$= \frac{1}{4}\sin 2x + \frac{1}{2}x + c$</td><td>1 – substitution
2 – correct integration</td></tr>
<tr><td>(d)</td><td>$y'' + y' - 2y = 0$
$k^2e^{kt} + ke^{kt} - 2e^{kt} = 0$
$e^{kt}(k^2+k-2) = 0$
$e^{kt}(k+2)(k-1) = 0$
$k = 1, -2$</td><td>1 – differentiation to find $y' and\ y''$
1 - substitution correct
1 – solutions</td></tr>
<tr><td>(e)</td><td>$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^4 x\, dx$
$u = \cos x$
$\frac{du}{dx} = -\sin x$
$\frac{du}{-\sin x} = dx$
$u = 0$ when $x = 0, u = \frac{1}{2}$</td><td>1 – substituting in for u and du
1 – changed the bounds or resubstituted after integration
1 – solution</td></tr>
</table>

	$$\int_0^{\frac{\pi}{3}} \sin x\,(1-\cos^2 x)\cos^4 x\,dx$$ $$\int_1^{\frac{1}{2}}(1-u^2)u^4\,du$$ $$=\left[\frac{u^5}{5}-\frac{u^7}{7}\right]_1^{\frac{1}{2}}$$ $$=\frac{1}{2^5\times 5}-\frac{1}{2^7\times 7}-\frac{1}{5}+\frac{1}{7}$$ $$=\frac{233}{4480}$$	
An alarming number of students integrated u^6 to be $\frac{u^6}{6}$. This is extension 1, you should know better.		
(f)	$$\frac{dy}{dx}=\sec y$$ $$\cos y\,dy = dx$$ $$\sin y = x + c$$ $$y=\sin^{-1}(x+c)$$ $$sub\ \left(\frac{1}{2},\frac{\pi}{6}\right)$$ $$\sin\frac{\pi}{6}=\frac{1}{2}+c$$ $$\frac{1}{2}=\frac{1}{2}+c$$ $$c=0$$ $$y=\sin^{-1}(x)$$	1 – finding $\sin y = x + c$ 2 – final answer
Generally this question was done well. However, when a question asks for a solution your final answer should be written as $y=f(x)$		

Question 6

a)	$$\int \frac{6+x}{4+x^2}dx$$ $$\int \frac{6}{4+x^2}+\frac{x}{4+x^2}dx$$ $$=3\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{2}\ln(4+x^2)+c$$ or $$3\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{2}\ln\left(1+\frac{x^2}{4}\right)+c$$	1 mark – splits the integral 2 marks – integrates correctly but leaves off a plus c or gets a constant wrong. 3 marks – correct with +c
(b)	$$\frac{dy}{dx}=xy$$ trivial solution $y=0$ $$\int\frac{1}{y}dy=\int x dx$$ $$\ln\lvert y\rvert = x^2+c$$	One mark for separating and integrating correctly two marks for $y=Ae^{x^2}$ defining A as a real number

	$\lvert y\rvert = e^c e^{x^2}$ $y = Ae^{x^2}, A\ is\ a\ real\ number$	
(c)	$;\dfrac{10t}{1+t^2} + \dfrac{4-4t^2}{1+t^2} = 5$ $10t + 4 - 4t^2 = 5 + 5t^2$ $9t^2 - 10t - 1 = 0$ $(9t-1)(t-1) = 0$ $t = 1\ or\ t = \dfrac{1}{9}$ $\tan\dfrac{x}{2} = 1$ $\dfrac{x}{2} = \dfrac{\pi}{4} \quad (0 \le \dfrac{x}{2} < \pi$ $x = \dfrac{\pi}{2}$ $\tan\left(\dfrac{x}{2}\right) = \dfrac{1}{9}$ $x = 2\tan^{-1}\left(\dfrac{1}{9}\right)$ $\pi\ is\ not\ a\ solution\ as\ the\ cofficient\ of$ $\cos x \neq 5$	1 –finds solutions in t 2 – Solves for x Most students found solutions in t, few went on to find the correct solutions in x
(d)	$\pi\int_0^{60} y + 50dy$ $= \pi\left[\dfrac{y^2}{2} + 50y\right]_0^{60}$ $= 4800\pi\ units^2$	1 – correct integral but left off π 2 – correct solution $done\ well$
ii)	$V = \pi\left(\dfrac{h^2}{2} + 50h\right) = 2800\pi$ $h^2 + 100h - 5600 = 0$ $h = 40, -140$ $h = 40, h > 0$	Correct answer Done well

Question 7

a)	SHOW	SHOW
ii)	Trivial solutions $P = 0, P = 100 - P$ $$\frac{1}{P(100-P)}dP = 0.001dt$$ $$\int \left(\frac{1}{100P} + \frac{1}{100(100-P)}\right) dP = \int 0.001dt$$ $$\frac{1}{100}(\ln\lvert P\rvert + \ln\lvert 100-P\rvert) = 0.0001t + c$$ $$\ln\left(\frac{\lvert P\rvert}{\lvert 100-P\rvert}\right) = 0.1t + c$$ $$\frac{\lvert P\rvert}{\lvert 100-P\rvert} = e^c e^{0.1t}$$ $$P = (100-P)Ae^{0.1t}, A\ is\ a\ real\ number$$ $$P + PAe^{0.1t} = 100Ae^{0.1t}$$ $$P(1 + Ae^{0.1t}) = 100Ae^{0.1t}$$ $$P = \frac{100Ae^{0.1t}}{1 + Ae^{0.1t}}$$ $$P = \frac{100}{\frac{1}{Ae^{0.1t}} + 1}$$ $$= \frac{100}{Be^{-0.1t} + 1}$$	1 mark off for no absolute value sign 1 mark for integrating 1 mark for getting up to $$P = \frac{100Ae^{0.1t}}{1 + Ae^{0.1t}}$$
b)	$$\tan(4x) = \frac{\cos x - \sin x}{\cos x + \sin x}$$ $$= \frac{-\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)}$$ $$= -\tan\left(x - \frac{\pi}{4}\right)$$ $$= \tan\left(\frac{\pi}{4} - x\right)$$ $$4x = \frac{\pi}{4} - x + n\pi$$ $$5x = \frac{\pi}{4} + \mathrm{n}\pi$$ let $n = 0$ $$5x = \frac{\pi}{4}$$ $$x = \frac{\pi}{20}$$	1 mark for getting to $4x = \frac{\pi}{4} - x$ or equivalent. 1 mark for solution Not many students got this question. Many students found $\frac{\pi}{4}$ as the smallest error.
(c)	$$\sin(x)\, y' + \cos(x) y = \tan x$$ *by observing the left hand side is the product rule and so will integrate to* $y \sin x$ $$y \sin x = \int \frac{\sin x}{\cos x} dx$$ $$y \sin x = -\ln(\cos x) + c$$ $$y = \frac{c - \ln(\cos x)}{\sin x}$$	1 – integrates LHS or equivalent merit 2 – correct solution

d)i)	$$\frac{\mathrm{dv}}{\mathrm{dh}} = -\frac{gR^2}{v(R+h)^2}$$ $$vdv = -gR^2(R+h)^{-2}$$ $$\frac{v^2}{2} = gR^2(R+h)^{-1} + C$$ $$v^2 = \frac{gR^2}{R+h} + c$$ sub (0,u) $$u^2 = gR + c$$ $$c = u^2 - gR$$ $$v^2 = u^2 - 2gR + \frac{2gR^2}{R+h}$$ $$v^2 = u^2 - \frac{2gR^2 + 2gRh - 2gR^2}{R+h}$$ $$v^2 = u^2 - \frac{2gRh}{R+h}$$ $$v^2 = u^2 - \frac{2gR}{\frac{R}{h}+1}$$	
ii)	$$\lim_{h\to\infty}\left(u^2 - \frac{2gR}{\frac{R}{h}+1}\right) > 0$$ $$u^2 - \frac{2gR}{1+0} > 0$$ $$u^2 > 2gR$$ $$u > \sqrt{2gR}$$ $$minimum\ launch\ velocity\ is$$ $$u = \sqrt{2gR}$$ as the limiting value of v will approach the limit from above	1 mark find u