Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Student Number

2023 **Year 12 Examination**

Mathematics Extension 1

Task 3

19/06/2023

Q	Marks
MC	/4
5	/12
6	/10
7	/8
8	/8
Total	/42

General Instructions

- Reading time – 5 minutes
- Working time – 55 minutes
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Show relevant mathematical reasoning and/or calculations
- No white-out may be used

Total Marks: 42

Section I - 4 marks

- Allow about 6 minutes for this section

Section II - 38 marks

- Allow about 49 minutes for this section

This question paper must not be removed from the examination room.

This assessment task constitutes 25% of the course.

Section I

4 marks
Attempt Questions 1–4
Allow about 6 minutes for this section.

Use the multiple-choice sheet for Question 1–4.

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1 Which expression gives the following indefinite integral?

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx$$

A. $\frac{1}{4}\arcsin\frac{2x}{3}+c$

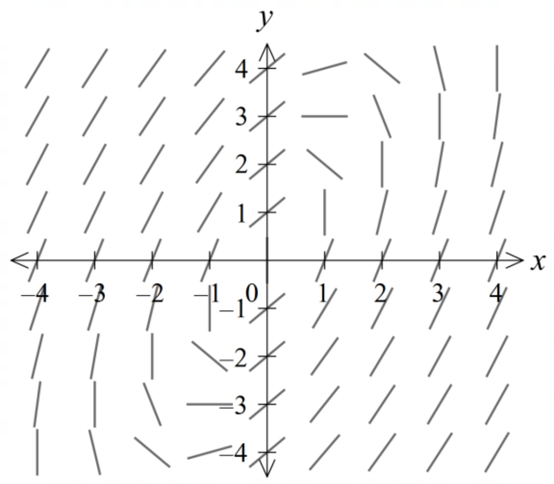
B. $\frac{1}{3}\arcsin\frac{2x}{3}+c$

C. $\frac{1}{2}\arcsin\frac{2x}{3}+c$

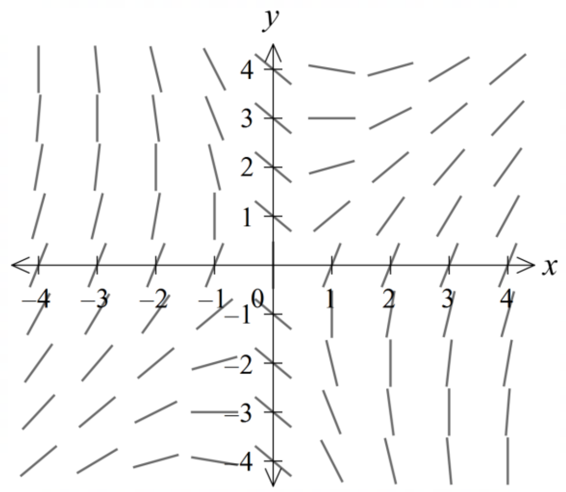
D. $\arcsin\frac{2x}{3}+c$

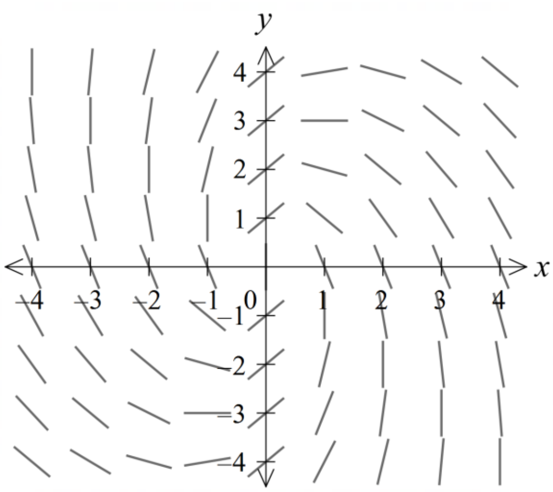
2 Which slope field represents the differential equation $y' = \frac{3x-y}{x+y}$?

A.



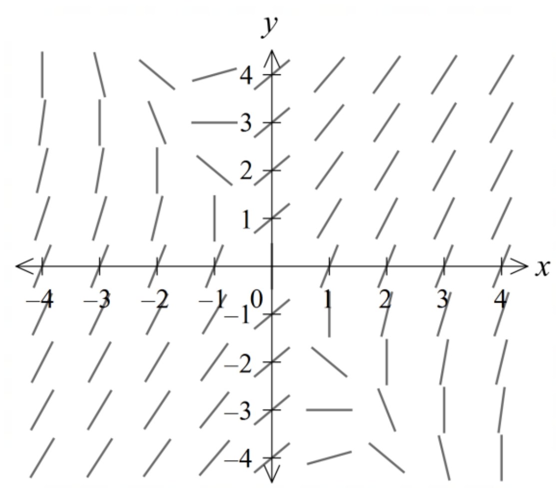
B.





C.

D.



3 What is the global maximum for the curve $y = 2\sin x + 6\cos x + \sqrt{10}$?

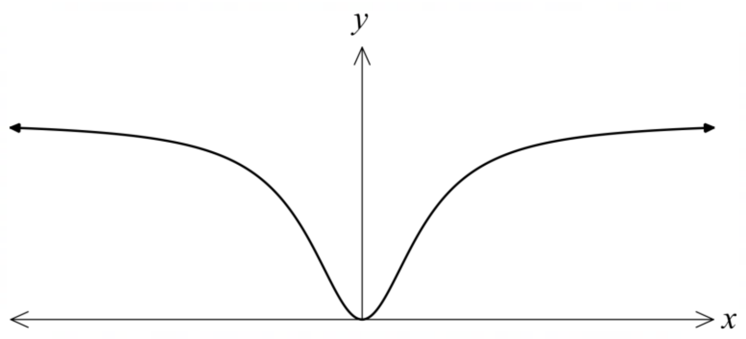
A. $y = 2\sqrt{2} + \sqrt{10}$

B. $y = 2\sqrt{10}$

C. $y = 4\sqrt{2} + \sqrt{10}$

D. $y = 3\sqrt{10}$

4 The graph of the function $f(x) = \frac{4x^2}{x^2+1}$ is shown below.



The derivative of the function is $f'(x) = \frac{8x}{(x^2+1)^2}$

When the function is defined for $x > 0$, an inverse function $y = f^{-1}(x)$ exists.

What is the derivative of the inverse function at the point $(3, \sqrt{3})$?

A. $\dfrac{2}{\sqrt{3}}$

B. $\dfrac{\sqrt{3}}{2}$

C. $\dfrac{6}{25}$

D. $\dfrac{25}{6}$

End of Section I

Section II

38 marks
Attempt Questions 5–8.
Allow about 49 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (12 marks) Answer each question in a NEW booklet.

(a) Differentiate the function **2**

$$f(x) = \arctan(x^3)$$

(b) Solve the equation $\cos 5x + \cos 3x = 0$ for $0 \leq x \leq \pi$. **2**

(c) Find the volume of revolution formed when the curve $y = \sin x$ is rotated about the x-axis between 0 and π. **2**

(d) Using the substitution $t = \tan\left(\frac{x}{2}\right)$, solve the following equation for $x \in [0, 360°]$, leaving your answers correct to the nearest degree. **3**

$$4\cos x + \sin x = 1$$

(e) Find the particular solution to the differential equation below which passes through the point $\left(\frac{1}{2}, 1\right)$. **3**

$$\frac{dy}{dx} = \frac{6}{y^2}$$

Question 6 (10 marks) Answer each question in a NEW booklet.

(a) (i) Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x-\alpha)$, where $\alpha \in [0, \pi]$. **2**

(ii) Hence, or otherwise, solve the following equation for $x \in [0, 4\pi]$. **2**

$$\sqrt{3}\sin x - \cos x = \sqrt{2}$$

(b) 40 elks were released into the Water Valley Game reserve by the state game commission. Assuming that the elk population, P, grows according to the logistic model as shown below with a growth constant of $k = \ln\left(\frac{11}{9}\right)$ per year.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right)$$

(i) What is the carrying capacity of elks in this game reserve? **1**

(ii) Given that **3**

$$\frac{1}{P(4000-P)} = \frac{1}{4000P} + \frac{1}{4000(4000-P)}$$

find the general solution to the differential equation above, leaving your answer in the form:

$$P = \frac{A}{1 + \frac{e^{-kt}}{C}}$$

(iii) At what time t is the population of elks growing the fastest? **2**

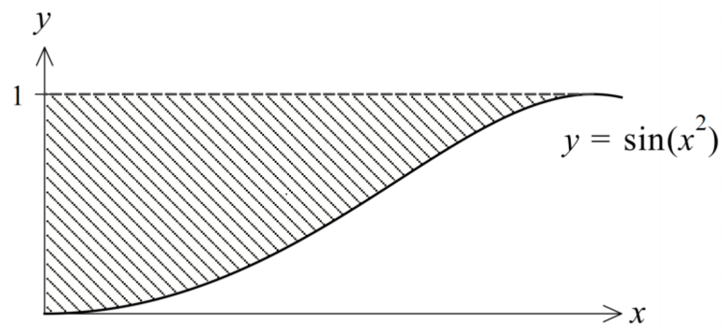
Question 7 (8 marks) Answer each question in a NEW booklet.

(a) (i) Find the derivative of the function $f(x) = x\sin^{-1}x$. **1**

(ii) By using the substitution $u = 1 - x^2$, show that **2**

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + c$$

(iii) The shaded area below is the region bounded by the curve $y = \sin(x^2)$, the y-axis and the curve $y = 1$. **3**



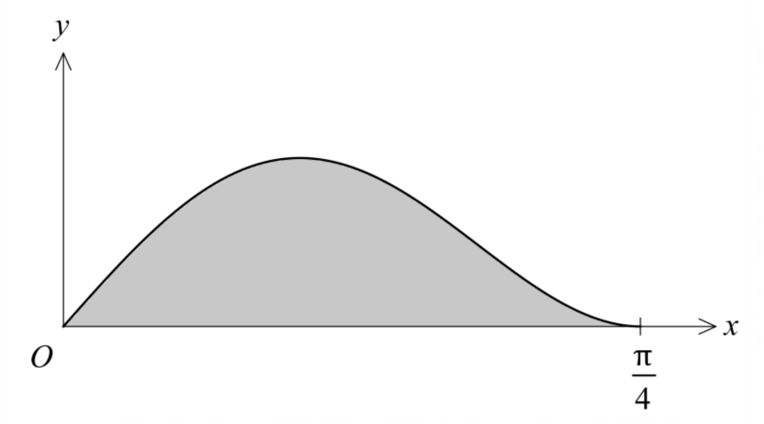
Find the volume formed when this area is rotated around the y-axis.

(b) Find the general solution to the differential equation, leaving your answer with y as the subject. **2**

$$(1+x^2)\frac{dy}{dx} = x^2e^{-y}$$

Question 8 (8 marks) Answer each question in a NEW booklet.

(a) The diagram shows the curve $y = \sin x \cos^2 2x$ for $0 \leq x \leq \frac{\pi}{4}$. **3**

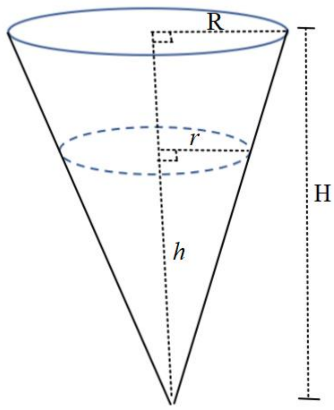


Using the substitution $u = \cos x,$ find by integration the exact area of the shaded region bounded by the curve and the x-axis.

(b) Experiments show that water leaks out through the vertex of a cone at a rate proportional to the square root of the height, h, of the water in the container.

A conical tank of height H units, and radius of the top R units stand on its vertex. The tank has a hole, and the water leaks out of the tank through the hole.

It takes exactly an hour to empty the full tank.



(i) Show that **2**

$$\frac{dh}{dt} = -\frac{kH^2}{\pi R^2 h^{\frac{3}{2}}}$$

(ii) If the cone is filled to the top, find the exact time it takes to reduce the water height level by half. **3**

End of paper

2023 Extension 1 Task 3 Marking scheme

1.	$\int \frac{1}{\sqrt{9-4x^2}}dx$ $Let\ u = 2x$ $du = 2dx$ $\frac{1}{2}\int \frac{1}{\sqrt{9-u^2}}du$ $\frac{1}{2}\sin^{-1}\frac{u}{3} + C$ $\frac{1}{2}\sin^{-1}\frac{2x}{3} + C$ C		
2	$x = 0, y' = -1$ B		
3.	$R + \sqrt{10}$ $R = \sqrt{4+36} = 2\sqrt{10}$ Max value $= 3\sqrt{10}$ D		
4.	We need to find the gradient of the $f^{-1}(x)$ at the point $(3, \sqrt{3})$. The corresponding point on $f(x)$ is $(\sqrt{3}, 3)$. $f'(x)$ at $x = \sqrt{3}$ is $\frac{8\sqrt{3}}{4^2} = \frac{\sqrt{3}}{2}$ Hence, the gradient of the inverse function is $\frac{2}{\sqrt{3}}$. A		

Question 5

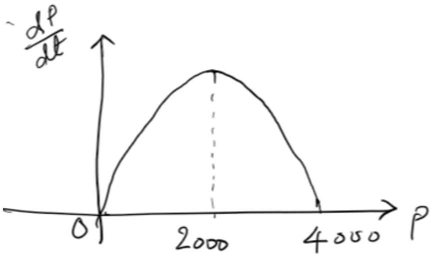
5a	$f(x) = \arctan(x^3)$ $f'(x) = \dfrac{3x^2}{1+(x^3)^2}$ $= \dfrac{3x^2}{1+x^6}$	2 marks: correct answer 1 mark: minor error (using sec^2 was not considered a minor error)	Well done across the cohort.
5b	$\cos 5x + \cos 3x = 0$ for $0 \le x \le \pi$. $2\cos 4x \,.\, \cos x = 0$ $\cos 4x = 0$ $4x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}, \ldots.$ $x = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}, \cancel{\dfrac{9\pi}{8}}, \ldots.$ $\cos x = 0$ $x = \dfrac{\pi}{2}$ So $x = \frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$	2 marks: gives the six correct solutions. 1 mark: correctly applies the $\cos A + \cos B$ formula and gives solves either $\cos x = 0$ or $\cos 4x = 0$ correctly.	Many students did this well, and were able to solve in the domain well. A section of the cohort could not recognise the sums-to-products identity and tried to expand with compound angles.
5c	Volume = $\displaystyle\int_0^{\pi} \pi y^2 dx$ $\displaystyle\int_0^{\pi} \pi sin^2 x\, dx$ $\dfrac{\pi}{2}\displaystyle\int_0^{\pi} 1 - \cos 2x \; dx$ $\dfrac{\pi}{2}\left[x - \dfrac{\sin 2x}{2}\right]_0^{\pi}$ $\dfrac{\pi}{2}[\pi - 0 + 0]$ $\dfrac{\pi^2}{2}$ units^3	1 mark: correctly writes the volume expression 1 mark: correctly integrates and gives the volume with correct units	Well done by many students, who understood how to calculate the volume and integrate $\sin^2 x$. Students needed to have correct units in this question
5d	$t = \tan\dfrac{x}{2}$ $4\cos x + \sin x = 1$ $4\left(\dfrac{1-t^2}{1+t^2}\right) + \dfrac{2t}{1+t^2} = 1$ $4 - 4t^2 + 2t = 1 + t^2$ $5t^2 - 2t - 3 = 0$ $(5t+3)(t-1) = 0$	5 elements that were looked for: *correctly applies the t-formula *correctly solves the quadratic equation *correctly solves for x for $t = -\frac{3}{5}$	Many students received 2 marks in this question. Most used the correct t-formulae and solving the equation from there.

	$t = -\frac{3}{5}$, $t = 1$ $\tan\frac{x}{2} = -\frac{3}{5}$ $\frac{x}{2} = 149°$ $x = 298°$ $\tan\frac{x}{2} = 1$ $\frac{x}{2} = 45°, \cancel{225°}$ $x = 90°$ Testing at asymptotes: $x = 180°$ $4\cos 180 + \sin 180 = -4 \neq 1$ $\therefore x = 180$ is not a solution. So $x = 90°$ and $298°$	* correctly solves for x for $t = 1$ * tests for solutions at the asymptote. 3 marks: All elements correct. 2 marks: at least 3 elements correct 1 mark: At least 2 elements correct.	A lot of students did not test the asymptote (when $x = 180°$).
5e	$\frac{dy}{dx} = \frac{6}{y^2}$ $\int y^2 dy = \int 6dx$ $\frac{y^3}{3} = 6x + C$ $y^3 = 18x + C$ 1 mark Substitute $\left(\frac{1}{2}, 1\right)$ $1 = 9 + C \Longrightarrow C = -8$ Then, $y = \sqrt[3]{18x - 8}$ 1 mark	1 mark: Correctly separates the variables 1 mark: Correctly integrates the functions for a general solution 1 mark: correctly substitutes the point and gives the particular solution.	Well done for the majority of the cohort. The most common mistake was from the very first step, where a number of students had $\int \frac{1}{y^2} dy$.

Question 6

6a(i)	$\sqrt{3}\sin x - \cos x \cong R\sin(x - \alpha)$ $= R\sin x\cos\alpha - R\cos x\sin\alpha$ $R\cos\alpha = \sqrt{3}$ $R\sin\alpha = 1$ $R = \sqrt{3+1} = 2$ $\tan\alpha = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$ Hence, the expression is $2\sin\left(x - \frac{\pi}{6}\right)$	2 marks: correct solution with full working 1 mark: Minor error (Award 1 mark if Correctly evaluates R Or Correctly evaluates α	
6a(ii)	Using (i),		

	$2\sin\left(x-\frac{\pi}{6}\right)=\sqrt{2}$ $\sin\left(x-\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}$ $x-\frac{\pi}{6}=\frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4},\frac{11\pi}{4},\frac{17\pi}{4}$ $x=\frac{5\pi}{12},\frac{11\pi}{12},\frac{29\pi}{12},\frac{35\pi}{4},\cancel{\frac{53\pi}{12}}$	2 mark: uses the result from (i), and solves the equation, giving all solutions 1 mark: Gives at least two solutions	
6b(i)	4000		
6b(ii)	$\frac{dP}{dt}=kP\left(1-\frac{P}{4000}\right)$ $\frac{dP}{dt}=\frac{kP(4000-P)}{4000}$ $\frac{dP}{P(4000-P)}=\frac{k}{4000}dt$ $\frac{1}{4000}\left[\frac{1}{P}+\frac{1}{4000-P}\right]dP=\frac{k}{4000}dt$ $t=0,\qquad P=40$ $\int_{40}^{P}\frac{1}{P}+\frac{1}{4000-P}dP=\int_{0}^{t}kdt$ $\left[\ln\left\|\frac{p}{4000-P}\right\|\right]_{40}^{P}=kt$ 1 mark $\ln\left[\frac{p}{4000-P}\times\frac{3960}{40}\right]=kt$ $\ln\left[\frac{99P}{4000-P}\right]=kt$ $\frac{99p}{4000-P}=e^{kt}$ $99P=4000e^{kt}-Pe^{kt}$ $P(99+e^{kt})=4000e^{kt}$ $P=\frac{4000e^{kt}}{99+e^{kt}}$ $P=\frac{4000}{99e^{-kt}+1}$ $P=\frac{4000}{1+\frac{e^{-kt}}{\frac{1}{99}}}$	1 mark: correctly separates the variables and integrates to get $\left[\ln\left\|\frac{p}{4000-P}\right\|\right]_{40}^{P}=kt$ 1 mark: correctly applies the initial conditions to get $\frac{99p}{4000-P}=e^{kt}$ 1 mark: correctly manipulates to find the value of C.	

	Thus, $C = \frac{1}{99}$		
6b (iii)	$\frac{dP}{dt}$ is the highest when, $\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right)$ $k = \ln\left(\frac{11}{9}\right) > 0$ $P\left(1 - \frac{P}{4000}\right)$ is maximum  Thus, $\frac{dP}{dt}$ is maximum, when $\therefore P = 2000$ From (ii), $P = \frac{4000}{99e^{-kt} + 1}$ $2000 = \frac{4000}{99e^{-\left(\ln\frac{11}{9}\right)t} + 1}$ $1 = \frac{2}{99e^{\left(\ln\frac{9}{11}\right)t} + 1}$ $1 = \frac{2}{99 \times \left(\frac{9}{11}\right)^{t} + 1}$ $1 + 99\left(\frac{9}{11}\right)^{t} = 2$ $99\left(\frac{9}{11}\right)^{t} = 1$ $\left(\frac{9}{11}\right)^{t} = \frac{1}{99}$ $t\left(\ln\frac{9}{11}\right) = \ln\frac{1}{99}$ $t = \frac{\ln\left(\frac{1}{99}\right)}{\ln\left(\frac{9}{11}\right)}$ $= 22.898 \ldots \ldots$	1 mark: Reasons why the growth rate is maximum occurs when P = 4000 and substitutes into the equation. 1 mark: solves the exponential function and evaluates the time correctly. (the exact value must be shown to get the full marks.)	

	$= 22.9$ years		

Question 7

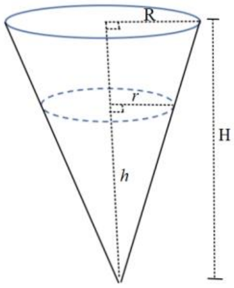
7a (i)	$y = x\sin^{-1}x$ $\frac{dy}{dx} = x \times \frac{1}{\sqrt{1-x^2}} + \sin^{-1}x \times 1$ $= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$	1 mark: Correctly applies the product rule.	Well done.
7a (ii)	$u = 1 - x^2$ $du = -2xdx$ $\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$ $-\frac{1}{2} \times 2\sqrt{u}$ $-\sqrt{u} + C$ $= -\sqrt{1-x^2} + C$	1 mark: correct conversions of the integrand in terms of u. 1 mark: correct integration and converts back to x to prove the result	Most students did this well. Question is asking to use the substitution, so you need to show the conversions of all components and must have an integrand just in terms of u.
7a (iii)	Volume about y-axis $= \int \pi x^2 dy$ $y = \sin x^2$ $x^2 = \sin^{-1} y$ V $= \pi \int_0^1 \sin^{-1} y\, dy$ 1 mark From (i) $\frac{d}{dx}(x\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$	1 mark: Derives the correct expression. $V = \pi \int_0^1 \sin^{-1} y\, dy$ To find volume 2 marks: correctly calculates the volume referring to the results from 7a(i) and (ii)	Most students got the first part and forming the volume expression, $V = \pi \int_0^1 \sin^{-1} y\, dy$ Very few students made the connection between parts (i) and (ii). Students need to pay particular attention to "differentiate, hence integrate" types of questions.

	$\int_0^1 \frac{d}{dx}(xsin^{-1}x)dx =$ $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx + \int_0^1 sin^{-1}x\,dx$ $\int_0^1 sin^{-1}x\,dx = [xsin^{-1}x]_0^1$ $-\left[-\sqrt{1-x^2}\right]_0^1$ (Using result from (ii)) $\text{Hence, Volume} = \pi\int_0^1 sin^{-1}x\,dx =$ $\pi\left(\left(\frac{\pi}{2}-0\right)+(0-1)\right)$ $= \pi\left(\frac{\pi}{2}-1\right)$ unit^3	Award only one mark: Minor error, but need to demonstrate working out the integral $\int_0^1 sin^{-1}x\,dx$ using the anti-derivative and evidence of using the result from (ii).	Also, please make sure that solutions to area, volume questions have units attached to it.
7b	$(1+x^2)\frac{dy}{dx} = x^2e^{-y}$ $\int e^y dy = \int \frac{x^2}{1+x^2}dx$ $\int e^y dy = \int \frac{1+x^2-1}{1+x^2}dx$ $\int e^y dy = \int 1 - \frac{1}{1+x^2}dx$ $e^y = x - \tan^{-1}x + C$ $y = \ln(x - \tan^{-1}x + C)$	1 mark: separates the variables and attempts to integrate (some attempt to integrate $\int \frac{x^2}{1+x^2}dx$ must be shown. 1 mark: correctly integrates and makes y subject.	Most students were able to separate the variables, and hence received 1 mark. Integrating the rational expression $\frac{x^2}{1+x^2}$: If the power of the numerator is the same or more than the denominator, you need to first divide, and then integrate each part. Very few students managed to do this.

Question 8

8a (i)	Shaded area =		

	$\text{Area} = \int_0^{\frac{\pi}{4}} \sin x \cos^2 2x \, dx$ Let $u = \cos x$ $du = -\sin x \, dx$ $\cos 2x = 2\cos^2 x - 1$ $\quad = 2u^2 - 1$ $x = 0, \quad u = 1$ $x = \frac{\pi}{4}, \quad u = \frac{1}{\sqrt{2}}$ 1 mark $\text{Area} = \int_0^{\frac{\pi}{4}} \sin x \cos^2 2x \, dx$ $= -\int_1^{\frac{1}{\sqrt{2}}} (2u^2 - 1)^2 du$ $= \int_{\frac{1}{\sqrt{2}}}^{1} 4u^4 - 4u^2 + 1 \, du$ $= \left[\frac{4u^5}{5} - \frac{4u^3}{3} + u\right]_{\frac{1}{\sqrt{2}}}^{1}$ $1\ mark$ $= \left(\frac{4}{5} - \frac{4}{3} + 1\right) - \left(\frac{4}{5} \times \frac{1}{4\sqrt{2}} - \frac{4}{3} \times \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$ $= \frac{7}{15} - \frac{1}{\sqrt{2}}\left(\frac{1}{5} - \frac{2}{3} + 1\right)$ $= \frac{7}{15} - \frac{8}{15\sqrt{2}}$ $= \frac{7 - 4\sqrt{2}}{15} \text{ unit}^2$ 1 mark	1 mark: uses the substitution and converts the integrand and the limits of integration in terms of u. 1 mark: correctly integrates. 1 mark: correct evaluation and gives the area in exact value in unit^2.	
7b (i)	$\frac{dV}{dt} = -k\sqrt{h}$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$	1 mark: develops the volume formula $V = \frac{\pi R^2 h^3}{3H^2}$ 1 mark: uses chain rule and equates to	

	 $V = \frac{1}{3}\pi r^2 h$ $\frac{r}{h} = \frac{R}{H}$ $r = \frac{Rh}{H}$ $V = \frac{1}{3}\pi\left(\frac{Rh}{H}\right)^2 h$ $V = \frac{\pi R^2 h^3}{3H^2}$ $\frac{dV}{dh} = \frac{\pi R^2 h^2}{H^2}$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\frac{\pi R^2 h^2}{H^2} \times \frac{dh}{dt} = -k\sqrt{h}$ $\frac{dh}{dt} = -\frac{kH^2}{\pi R^2 h^{\frac{3}{2}}}$	$\frac{dV}{dt} = -k\sqrt{h}$ to derive the result	
	$h^{\frac{3}{2}}\,dh\ -\frac{kH^2}{\pi R^2}\,dt$ Integrating both sides, $\int_H^0 h^{\frac{3}{2}}\,dh = \frac{-kH^2}{\pi R^2}\int_0^1 dt$ $\frac{-2H^{\frac{5}{2}}}{5} = \frac{kH^2}{\pi R^2}$ $k = \frac{2\pi R^2\sqrt{H}}{5}$ Hence, $h^{\frac{3}{2}}\,dh = \frac{2\pi R^2\sqrt{H}\,H^2}{5\pi R^2}\,dt$ $h^{\frac{3}{2}}\,dh = \left(-\frac{2\,H^{\frac{5}{2}}}{5}\right)dt$ Let T be the time taken to get the height $\frac{H}{2}$.	1 mark: evaluates k 1 mark: rewrites the $\frac{dh}{dt}$ expression. $h^{\frac{3}{2}}\,dh = \left(-\frac{2\,H^{\frac{5}{2}}}{5}\right)dt$ 1 mark: Integrates to evaluate the time taken to reduce the height to half.	

	$\int_{H}^{\frac{H}{2}} h^{\frac{3}{2}}\, dh = \int_{0}^{T} \left(-\frac{2\, H^{\frac{5}{2}}}{5}\right) dt$ $\left[\frac{2h^{\frac{5}{2}}}{5}\right]_{H}^{\frac{H}{2}} = -\frac{2H^{\frac{5}{2}}\, T}{5}$ $\frac{2H^{\frac{5}{2}}}{5}\left[\left(\frac{1}{2}\right)^{\frac{5}{2}} - 1\right] = -\frac{2H^{\frac{5}{2}}}{5}\, T$ $T = 1 - \left(\frac{1}{2}\right)^{\frac{5}{2}} = 0.8\, hours$		