Mathematics Extension 1

Year 11 Assessment Task 2

Week 4 Term 3 2021

Name: ....

Teacher: ....

Thursday 5th August 2021

General Instructions

- All answers are to be written on your own lined paper so they can be scanned and uploaded as a single PDF file to your teacher's google classroom. Make sure that you write your name on each of the lined pages used.

- **Start each question on a new page.**

- Calculators approved by NESA may be used.

- Write using black pen.

- In Questions 1– 5, show relevant mathematical reasoning and/or calculations. Marks may be deducted for insufficient or illegible work.

- For any multiple-choice question, **write the letter** that corresponds to your answer on your writing paper.

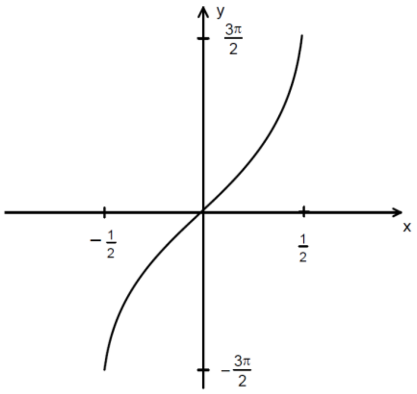
- Time Allowed: **55 minutes inclusive of task downloading then answering on your own lined paper, scanning and uploading a single PDF file for your responses to your teacher's google classroom.**

- **Total marks – 32**

Setter: YUS

Question 1 (6 marks)

(a) Which of the alternatives could represent the inverse trigonmonetric function drawn below?



A. $y = \frac{1}{3}\sin^{-1}\frac{x}{2}$

B. $y = \frac{1}{3}\sin^{-1}2x$

C. $y = 3\sin^{-1}\frac{x}{2}$

D. $y = 3\sin^{-1}2x$

(b) If $\log_5 8 = a$, prove that $\log_{10} 2 = \frac{a}{a+3}$. **2**

(c) An inverted cone of radius r = 40 cm and depth h = 70 cm is filled with water. The water flows out of a hole in the apex of the cone at a constant rate of 30π cm^3/minute.

(i) Show that the volume V of the cone is given by $V = \frac{16\pi h^3}{147}$. **1**

(ii) Find the rate at which the water level is decreasing when the depth of water is 10cm. **2**

End of Question 1

Question 2 (6 marks) Start a new page.

A baked cake cools at a rate proportional to the difference between the cake temperature (T) and the room temperature (20ºC). Initially the cake is 80ºC and after 10 minutes the cake is 50ºC.

(i) Show that $T = 20 + Ae^{kt}$ satisfies the cooling equation $\dfrac{dT}{dt} = k(T - 20)$. **1**

(ii) Show that $A = 60$. **1**

(iii) Find the exact value of k and hence find when the cake temperature reaches 35°C. **2**

(iv) Find the limiting temperature of the cake. **1**

(v) What is the rate of cooling after exactly 10 minutes? **1**

End of Question 2

Question 3 (7 marks) Start a new page.

(a) A driver takes 3 hours to travel the distance between two points P and Q on a country road. At time t hours after passing P, the driver's speed v km/h is given by $v = 70 + 40e^{-t}$.

(i) Calculate the speed when the driver passes point Q, to the 2 significant figures. **1**

(ii) Write the acceleration in terms of t. **1**

(iii) Sketch the velocity-time curve and comment on the velocity for large values of t. **2**

(b) Find the value of $\tan\left(\dfrac{7\pi}{12}\right)$ as a simplified surd with a rational denominator. **3**

End of Question 3

Question 4 (6 marks) Start a new page.

(a) Using the *t*-substitutions, or otherwise, prove the identity **3**

$$\frac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} = \tan^2\theta$$

(b)

For the function $f(x) = \begin{cases} \dfrac{1}{x} & \text{for } 0 < x < 2 \\ \dfrac{1}{2} & \text{for } x = 2 \\ 1 - \dfrac{1}{4}x & \text{for } x > 2 \end{cases}$

(i) Show that the curve is continuous at $x = 2$. **1**

(ii) Determine whether the function is differentiable at $x = 2$ and if so, state the value of $f\,'(2)$. Explain your reasoning carefully. **2**

End of Question 4

Question 5 (7 marks) Start a new page.

(a) Determine where the curve $y = \sin 5x + \sin x$ cuts the *x*-axis for $0 \le x \le \dfrac{\pi}{2}$. **2**

(b) Consider the function $y = \tan^{-1}(\tan x)$.

(i) State its domain and whether it is even, odd or neither **2**

(ii) Simplify $\tan^{-1}(\tan x)$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. **1**

(iii) Use the above information and a table of values if necessary to sketch the function for $-\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$. You may assume $\tan x = \tan(x+\pi)$. **2**

End of paper

Year 11 Mathematics Extension 1 AT2 SOLUTIONS

Suggested Solution (s)	Comments

Question 1

a) $-\frac{1}{2} \le x \le \frac{1}{2}$

$-1 \le 2x \le 1$

& $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

$-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

(1)

So $\frac{y}{3} = \sin^{-1} 2x$

ie) $y = 3\sin^{-1} 2x$

b) $\log_5 8 = a \Rightarrow \log_5 2^3 = a$

ie) $\log_5 2 = \frac{a}{3}$ (*)

Now $\log_{10} 2 = \frac{\log_5 2}{\log_5 10}$

$= \frac{\log_5 2}{\log_5 (2\times5)}$ or ✓

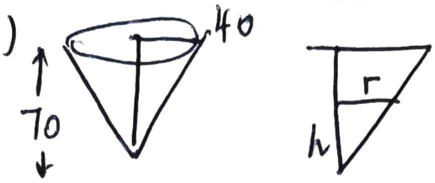
$= \frac{\log_5 2}{\log_5 2 + \log_5 5}$

$\therefore = \frac{a/3}{\frac{a}{3}+1}$ using (*)

$= \frac{a/3}{a/3 + 3/3}$ ✓

$= \frac{a}{a+3}$

c)



i) $V = \frac{1}{3}\pi r^2 h$ but $\frac{r}{h} = \frac{40}{70}$

$r = \frac{4h}{7}$

$\therefore V = \frac{1}{3}\pi\left(\frac{4h}{7}\right)^2 h$ ✓

$= \frac{16\pi h^3}{147}$

ii) Now $\frac{dV}{dh} = \frac{48\pi h^2}{147}$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \frac{147}{48\pi h^2} \times -30\pi$ ✓

Now $h = 10$

$\frac{dh}{dt} = \frac{147 \times -30\pi}{48\pi(10)^2}$

$= \frac{-4410}{4800}$ ✓

$= -0.91875$ cm/min.

$\therefore$ The depth of water is decreasing at 0.92 cm/min.

Comments: or $-\frac{147}{160}$ cm/min

(1)

Year 11 Mathematics Extension 1 SOLUTIONS

Suggested Solution (s)		Comments
Question 2 (i) $T = 20 + Ae^{kt}$ $LHS = \frac{dT}{dt} = Ae^{kt} \times k$ $= Ake^{kt}$ but $Ae^{kt} = T - 20$ $\therefore \frac{dT}{dt} = k(T-20)$ $= RHS$ (ii) When $t = 0$ $T = 80$ ie) $80 = 20 + Ae^0$ $A = 60$. (iii) when $t = 10$, $T = 50$ ie) $50 = 20 + 60e^{10k}$ $\frac{30}{60} = e^{10k}$ $\ln\frac{1}{2} = 10k$ $-\ln 2 = 10k$	When $T = 35$, $t = ?$ ie) $35 = 20 + 60e^{kt}$ where $k = \frac{-\ln 2}{10}$ $\frac{15}{60} = e^{kt}$ $\ln\frac{1}{4} = kt$ $t = \frac{\ln\frac{1}{4}}{k}$ $t = 20\text{min}$. iv) As $t \to \infty$ $60e^{kt} \to 0$ $\therefore T \to 20$ The limiting temperature is $20°$. v) $\frac{dT}{dt} = k(T-20)$ $= k(50-20)$ where $k = -\frac{\ln 2}{10}$ $= -2.079441$. The rate of cooling is $2.079°C/\text{min}$	

$\therefore k = -\frac{\ln 2}{10}$

Year 11 Mathematics Extension 1 SOLUTIONS

(3)

Suggested Solution (s)		Comments

Question 3

a) $V = 70 + 40e^{-t}$

i) For Q, $t = 3$

$\therefore V = 70 + 40e^{-3}$

$= 71.99$

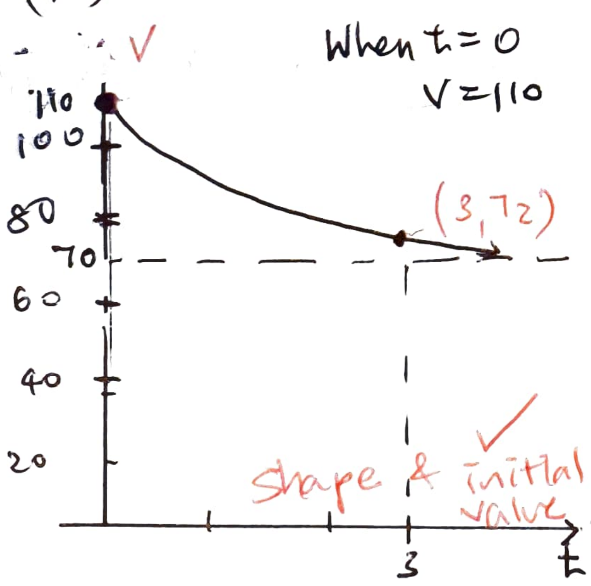
$\doteqdot 72 \text{ km/h}$ ✓

ii) $a = -40e^{-t}$ ✓

(iii)

When $t = 0$

$V = 110$



As $t \to \infty$, $40e^{-t} \to 0$

$\therefore V \to 70$

as t increases the velocity approaches 70 km/h ✓

(b) $\tan \frac{7\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

ie) $\tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{3}} \checkmark$$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \checkmark$$

$$= \frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{1 + 2\sqrt{3} + 3}{-2}$$

$$= -2 - \sqrt{3} \checkmark$$

Year 11 **Mathematics Extension 1** **SOLUTIONS**

Suggested Solution (s)	Comments

Question 4

Method I

Let $\tan\theta = t$

$$LHS = \frac{\frac{2t}{1-t^2} - t}{\frac{2t}{1-t^2} + \frac{1}{t}} \qquad RHS = t^2$$

$$= \frac{2t - t(1-t^2)}{1-t^2} \div \frac{2t^2 + (1-t^2)}{t(1-t^2)} \checkmark$$

$$= \frac{2t - t + t^3}{\cancel{(1-t^2)}} \times \frac{t\cancel{(1-t^2)}}{2t^2+1-t^2} \checkmark$$

$$= (t^3+t) \times \frac{t}{t^2+1} \checkmark$$

$$= t(\cancel{t^2+1}) \times \frac{t}{\cancel{(t^2+1)}}$$

$$= t^2$$

$$= RHS$$

Method II

$$\frac{\frac{2\tan\theta}{1-\tan^2\theta} - \frac{\tan\theta}{1}}{\frac{2\tan\theta}{1-\tan^2\theta} + \frac{1}{\tan\theta}} \checkmark$$

$$= \frac{(\tan\theta + \tan^3\theta)/(1-\tan^2\theta)}{(\tan^2\theta + 1)/\tan\theta(1-\tan^2\theta)} \checkmark$$

$$= \frac{\tan\theta(\cancel{1+\tan^2\theta})}{\cancel{(1-\tan^2\theta)}} \times \frac{\tan\,\cancel{(1-\tan^2\theta)}}{\cancel{(\tan^2\theta+1)}} \checkmark$$

$$= \tan^2\theta$$

2021 Year 11 Mathematics Extension 1 Task 2 SOLUTIONS

Suggested Solution (s)	Comments

Question 4 (b)

(i) Given $f(2) = \frac{1}{2}$, we require $\lim\limits_{x \to 2^-} f(x) = f(2) = \lim\limits_{x \to 2^+} f(x)$.

Now, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \frac{1}{x}$ or $\frac{1}{2}$ & $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left(1 - \frac{1}{4}x\right)$

$= 1 - \frac{1}{4} \times 2$ or $\frac{1}{2}$.

As $\lim\limits_{x \to 2^-} f(x) = f(2)$ or $\frac{1}{2} = \lim\limits_{x \to 2^+} f(x)$, f is cts at $x = 2$

2 marks for argument.

(ii) We require $\lim\limits_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$ & $\lim\limits_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$ to exist with $\lim\limits_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$

Now, $\lim\limits_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$ & $\lim\limits_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \to 0^-} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \qquad = \lim_{h \to 0^+} \frac{1 - \frac{1}{4}(2+h) - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0^-} \left(\frac{2 - (2+h)}{2(2+h)} \times \frac{1}{h}\right) \qquad = \lim_{h \to 0^+} \frac{1 - \frac{1}{2} - \frac{1}{4}h - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0^-} \frac{-h}{2h(2+h)} \qquad = \lim_{h \to 0^+} \frac{-\frac{1}{4}h}{h}$$

$$= \lim_{h \to 0^-} \frac{-1}{2(2+h)} \qquad = \lim_{h \to 0^+} -\frac{1}{4}$$

$$= -\frac{1}{4} \qquad = -\frac{1}{4}.$$

As $\lim\limits_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$, conclude that the function is differentiable at $x = 2$ with $f'(2) = -\frac{1}{4}$.

Suggested Solution (s)	Comments

Question 5

a) $y = \sin 5x + \sin x$

To cut x-axis

$\sin 5x + \sin x = 0$

$\sin(3x+2x) + \sin(3x-x) = 0$

ie) $2\sin 3x \cos 2x = 0$ ✓

So $\sin 3x = 0$ or $\cos 2x = 0$

$3x = 0, \pi, 2\pi, 3\pi \ldots$

or $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ ✓

$\therefore x = 0, \frac{\pi}{3}$ or $x = \frac{\pi}{4}$

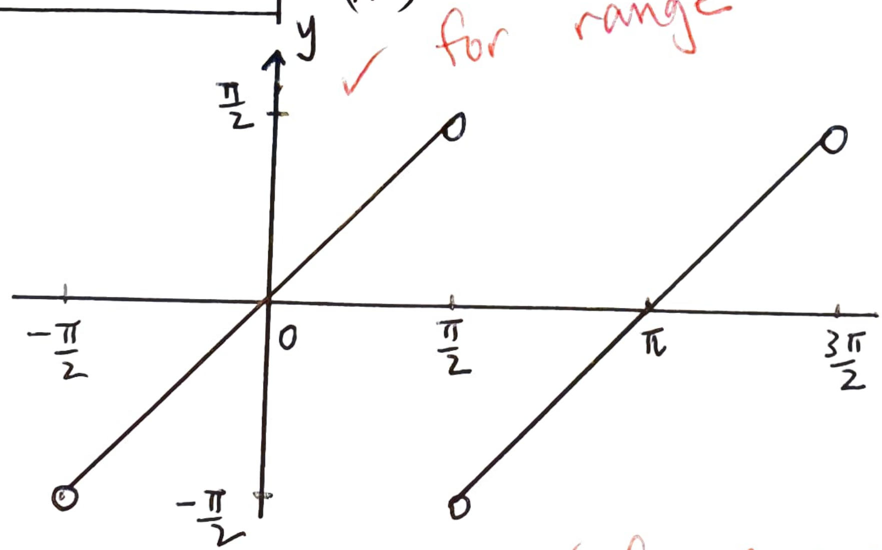
(b) For $y = \tan^{-1}(\tan x)$

(i) Domain : all real x except $x = \frac{(2n+1)\pi}{2}$ ✓ where n is an integer

& Odd function ✓

(ii) $\tan^{-1} \tan x = x$ ✓

(iii) ✓ for range



✓ for 2 straight line

∴ Carried error for 'open circle' from (i)