Ext 1

# NORTH SYDNEY BOYS HIGH SCHOOL

**2010 Year 11**
**Assessment task 3**

# Mathematics
## Extension 1

### General Instructions

- Working time – **53 minutes**
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

**Class Teacher:**
(Please tick or highlight)

- O Mr Berry
- O Mr Ireland
- O Mr Fletcher
- O Mr Lam
- O Mr Rezcallah
- O Mr Trenwith
- O Mr Weiss

Student Name:____________________________

(To be used by the exam markers only.)

| Question No | 1 | 2 | 3 | 4 | Total | Total % |
|---|---|---|---|---|---|---|
| Mark | /12 | /14 | /14 | /17 | /57 | /100 |

**Question 1 (12 marks)**

(a) Differentiate the following functions using the most appropriate method.

(i) $3x^5 + \dfrac{1}{x^2}$ 2

(ii) $(13 + x^2)^7$ 2

(iii) $x\sqrt{1 - x^2}$ 2

(b) Find

(i) $\lim\limits_{x \to 0} \dfrac{x}{x^2 + 10}$ 1

(ii) $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 4x}$ 2

(c) The tangent to the curve $y = 2x^3 + x^2$ at $x = 1$ cuts the $x$-axis at $(a, 0)$.
Find the equation of this tangent and the value of $a$. 3

**Question 2 (14 marks) Start a new page.**

(a) Differentiate from first principles: $y = f(x) = 5x^2 - 2x + 3$. 3

(b) For what values of $x$ is the curve $y = x^3 - x^2 - 8x + 1$ decreasing? 3

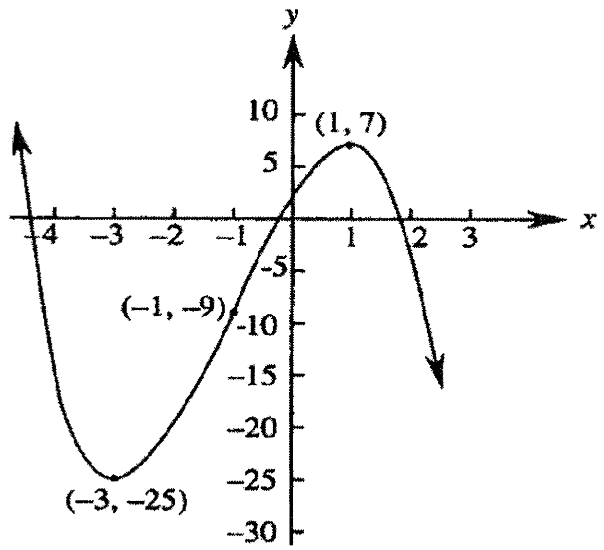
(c) The size of classes ($S$) at a local public school is decreasing and the rate at which this is happening is decreasing.

i) Determine the sign of $\dfrac{dS}{dt}$ and $\dfrac{d^2S}{dt^2}$. 2

ii) Draw a graph of $S$ versus $t$ to show this information. 1

(d) For what values of $b$ is the quartic $y = x^4 - bx^3 + 6x^2 + 13x - 7$ concave up at the point where $x = 2$? 3

(e) The curve $y = f(x)$ is shown below. The stationary points and point of inflexion are clearly shown. Sketch the graph of $y = f'(x)$ showing the $x$ values of all important features. 2



**Question 3 (14 marks) Start a new page.**

(a) The surface area in cm$^2$ of a balloon being inflated is given by $S = t^3 - 2t^2 + 7t + 8$ where $t$ is time in seconds. Find the rate of increase in the balloon's surface area after 5 seconds. 2

(b) A particle moves in a straight line so that its displacement $x$ metres from a fixed point O at time t seconds is given by $x = 3t^2 - 12t + 18$

(i) Find its velocity and acceleration as a function of time. 2

(ii) Find the initial velocity of the particle. 1

(iii) Find when and where the particle is at rest. 2

(iv) Find the distance travelled by the particle in the first 3 seconds. 2

(c) Consider the function:

$$f(x) = \begin{cases} \dfrac{1}{x} + 1 & x \leq -1 \\[2ex] \dfrac{x^2 - 1}{x - 1} & -1 < x < 2, \quad x \neq 1. \end{cases}$$

(i) Is the function continuous at $x = -1$? Justify your answer. 2

(ii) Is the function differentiable at $x = -1$? Justify your answer. 3

**Question 4 (17 marks) Start a new page.**

(a) Find the point of inflexion on the curve $y = 2x^3 - 12x^2 + 5$ 3

(b) If $y = f(x) = \dfrac{1+x^2}{1-x^2}$

(i) Find any intercepts of the graph $y = f(x)$ with the $y$-axis. 1

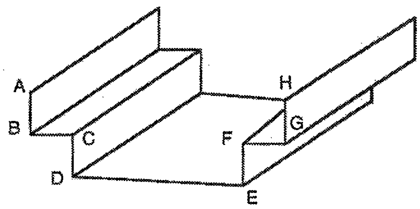
(ii) Find $\lim\limits_{x \to \infty} \left( \dfrac{1+x^2}{1-x^2} \right)$ 1

(iii) State the equation(s) of any asymptotes. 2

(iv) Find the position and nature of any turning point(s). 3

(v) Sketch the graph of $y = f(x)$ clearly showing all the above information 2

(b) A metal gutter is to be folded as shown, from a strip 60 cm wide (i.e. The total distance ABCDEFGH is 60 cm.) AB, BC, CD, EF, FG and GH are each $x$ cm and DE is $y$ cm. All angles are right angles.




(i) Write a relation between $x$ and $y$. 1

(ii) Find the area of the cross section (DEFGHABCD) of the gutter in terms of $x$. 2

(iii) Find the dimensions of the gutter if it is to have a maximum cross sectional area. 2

Date

| Solution of 2010 Yr11 Task 3 - Ext 1 | Marks. |
|---|---|
| **Question 1:** | |
| (a) (i) $y' = 15x^4 - 2x^{-3}$ <br> $= 15x^4 - \dfrac{2}{x^3}$. | (i) ✓✓ |
| (ii) $7(13+x^2)^6(2x)$ <br> $= 14x(13+x^2)^6$. | (ii) ✓✓ for correct rule. <br> 0 for only $7(13+x^2)^6$. |
| (iii) $1(\sqrt{1-x^2}) + x \cdot (-2x)\overset{\checkmark}{\tfrac{1}{2}}(1-x^2)^{-\frac{1}{2}}$. <br> $= \sqrt{1-x^2} \quad - \dfrac{x^2}{\sqrt{1-x^2}}$. <br> $= \dfrac{1-x^2-x^2}{\sqrt{1-x^2}} = \dfrac{1-2x^2}{\sqrt{1-x^2}}$. | (iii) ✓ for $\dfrac{-x}{\sqrt{1-x^2}}$. <br> ✓ for correct product rule. |
| (b) (i) $\lim\limits_{x\to 0} \dfrac{x}{x^2+10} = \dfrac{0}{10} = 0$. | b) (i) ✓ |
| (ii) $\lim\limits_{x\to 4} \dfrac{(x-4)(x+4)}{x(x-4)} = \lim\limits_{x\to 4} \dfrac{x+4}{x}$ <br> $= \dfrac{8}{4} = 2$. | (ii) ✓ for method. <br> ✓ for 2. |
| (c) $y' = 6x^2 + 2x$. <br> $m = 6(1)^2 + 2(1) = 8$. <br> $x=1, \ y = 2(1)^3 + 1^2 = 3$. <br> $y - 3 = 8(x-1)$ <br> $y = 8x - 5$. or $8x - y - 5 = 0$. <br> at $\cancel{y}\,y = 0$ : $8x - 5 = 0$. <br> $x = 5/8 = a$. | c) <br> ✓ for $m=8$. <br> ✓ correct line equation any f <br> ✓ for $a = 5/8$. |
| **Question 2:** | (a) |
| a) First method: <br> $f(x+h) = 5(x+h)^2 - 2(x+h) + 3$ <br> $= 5x^2 + 10xh + 5h^2 - 2x - 2h + 3$. | (Take away 1 if $\lim\limits_{h\to 0}$ is not writte |
| $y' = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ | ✓ correct rule |
| $= \lim\limits_{h\to 0} \dfrac{\cancel{5x^2} + 10xh + 5h^2 - \cancel{2x} - 2h + \cancel{3} - \cancel{5x^2} + \cancel{2x} - \cancel{3}}{h}$. | ✓ correct method |
| $= \lim\limits_{h\to 0} \dfrac{\cancel{h}(10x + 5\underset{0}{h} - 2)}{\cancel{h}} = 10x - 2$. | ✓ answer |

Solution of 2010 Yr 11 Task 3 - Ext 1

Question 2 Cont'd:

2ND method: $y' = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}$

$$y' = \lim_{u \to x} \frac{5u^2 - 2u + 3 - (5x^2 - 2x + 3)}{(u - x)}$$

$$= \lim_{u \to x} \frac{5(u^2 - x^2) - 2(u - x) + 3 - 3}{(u - x)}$$

$$= \lim_{u \to x} \frac{5(u+x)\cancel{(u-x)} - 2\cancel{(u-x)}}{\cancel{(u-x)}}$$

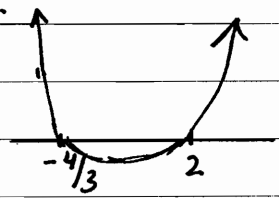
$$= \lim_{u \to x} \left[ 5(u+x) - 2 \right] = 5 \times 2x - 2 = 10x - 2.$$

(b) $y = x^3 - x^2 - 8x + 1.$

$y' = 3x^2 - 2x - 8 < 0.$

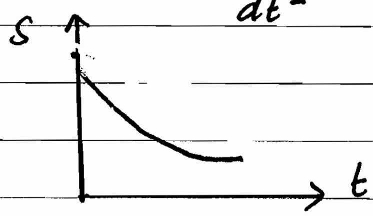
$(3x + 4)(x - 2) < 0.$

$-4/3 < x < 2.$




(c) (i) $\frac{dS}{dt} < 0$  $\frac{d^2S}{dt^2} > 0.$

(ii)




(d) $y' = 4x^3 - 3bx^2 + 12x + 13.$

$y'' = 12x^2 - 6bx + 12.$

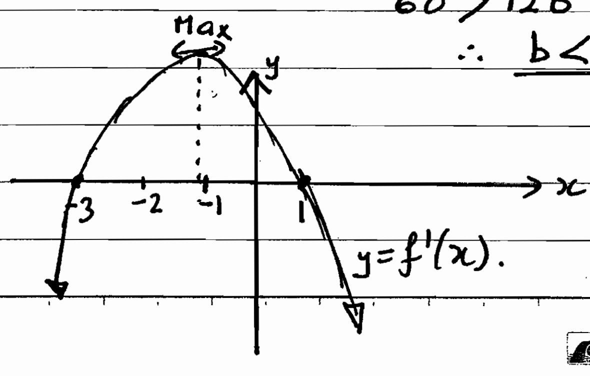
at $x = 2$: $12(2)^2 - 6b(2) + 12 > 0.$

$60 - 12b > 0.$

$60 > 12b$

$\therefore \underline{b < 5}$

(e)




Marks.

or 2ND method.

✓ for correct principle with $\lim_{u \to x}$

✓ for correct method.

✓ for correct answer.

(b)

✓ for correct $y' < 0$

✓ for correct factors.

✓ for correct inequality.

(c)

(i) ✓ for $\frac{dS}{dt} < 0$

✓ for $\frac{d^2S}{dt^2} > 0$

(ii) ✓ for correct shape of graph

d)

✓ for correct $y'$

✓ for $60 - 12b > 0$

✓ for $b < 5$.

(e)

✓ for correct shape.

✓ for all correct $x$ values clearly shown $x = -3, -1, 1.$

Solution of 2010 Yr11 Task 3 - Ext1

Marks.

Question 3.

Q3 -

(a) $S = t^3 - 2t^2 + 7t + 8.$

$\frac{dS}{dt} = 3t^2 - 4t + 7.$ ✓

$= 3(5)^2 - 4(5) + 7.$

$= 62 \text{ cm}^2/\text{s}.$ ✓ for $62\text{cm}^2/\text{s}$.

(b) $x = 3t^2 - 12t + 18.$

(i) $\frac{dx}{dt} = 6t - 12.$ (i) ✓ for $6t - 12$

$\frac{d^2x}{dt^2} = 6 \text{ m/s}^2.$ ✓ for $6\text{m/s}^2$

(ii) at $t = 0$ $v = 6(0) - 12 = -12 \text{ m/s}.$ (ii) ✓ for $-12$m/s (not 12m/s)

(iii) $v = 6t - 12 = 0.$

$6t = 12 \quad \therefore t = 2s.$ (iii) ✓ $t = 2s$.

$x = 3(2)^2 - 12(2) + 18.$

$x = 6m.$ ✓ for $x = 6m$

(iv)

| $t$ | 0 | 2 | 3 |
|---|---|---|---|
| $x$ | 18 | 6 | 9 |

Distance travelled $0 \le t \le 2$: $x = 12m.$

$2 \le t \le 3$ $x = 3m.$

total distance: $12 + 3 = 15m.$

(iv) ✓ for showing the 2 distan[ce] concept. ✓ for 15m.

(c) students who write only: yes, no should only get 1 mark if both answers correct.

(c)

(i) $f(x) = \begin{cases} \frac{1}{x} + 1 & x \le -1. \\ \frac{(x-1)(x+1)}{(x-1)} = x+1, & -1 < x < 2, x \ne 1. \end{cases}$

at $x = -1$: $\lim\limits_{x \to -1^-} f(x) = \frac{+1}{-1} - 1 = 0.$

$\lim\limits_{x \to -1^+} f(x) = -1 + 1 = 0.$

✓ for correct working or even correct sketch.

$\therefore$ Continuous at $x = -1$ since $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x).$ ✓ for continuity.

(ii) $f'(x) = \begin{cases} -\frac{1}{x^2} \\ 1 \end{cases}$

(ii) (3 marks)

$\lim\limits_{x \to -1^-} f'(x) = -\frac{1}{(-1)^2} = -1.$ $\lim\limits_{x \to -1^+} f'(x) = 1.$ ✓

✓ for each correct derivative

$\therefore$ Not differentiable at $x = -1$ as $\lim\limits_{x \to -1^-} f'(x) \ne \lim\limits_{x \to -1^+} f'(x).$

✓ for not differentiable with the reason.

Solution of Ext1 Task 3 - 2010.

Question 4.

a) $y = 2x^3 - 12x^2 + 5.$

$y' = 6x^2 - 24x.$

$y'' = 12x - 24 = 0.$

$x = 2.$

test inflexion

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $y''$ | $-12$ | 0 | $+12$ |
| | concave down | | concave up. |

$y = 2(2)^3 - 12(2)^2 + 5$

$= -27.$ $\therefore$ $(2, -27)$ is a point of inflexion since concavity changes.

(b) (i) at $x=0$, $y=1$. $(0,1)$

(ii) $\lim_{x\to\infty} \frac{1+x^2}{1-x^2} = -1.$

(iii) $x = \pm 1$

$y = -1$

(iv) $y' = \frac{2x(1-x^2) - (-2x)(1+x^2)}{(1-x^2)^2}.$

$= \frac{2x - \cancel{2x^3} + 2x + \cancel{2x^3}}{(1-x^2)^2}.$

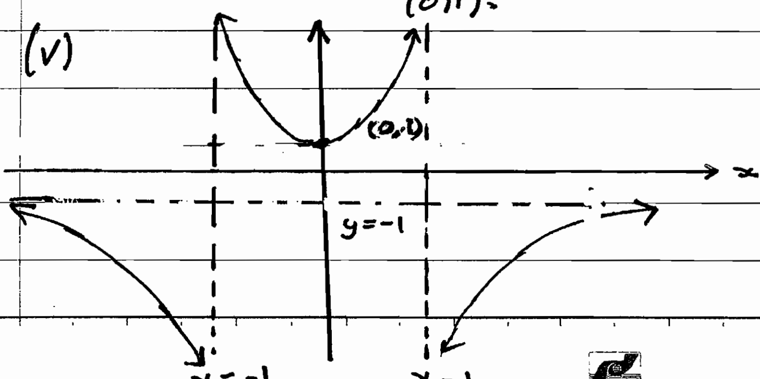
$= \frac{4x}{(1-x^2)^2}.$

$y' = 0.$ $\therefore$ $x = 0$, $y = 1$ $(0,1)$

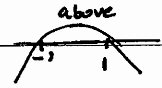
test:

| | $-2$ | 0 | 2 |
|---|---|---|---|
| $y' = \frac{4x}{(1-x^2)^2}$ | $-$ | 0 | $+$ |
| | ↘ | MIN $(0,1)$. | ↗ |

(v)




sign diagram:
$(1+x^2)(1-x)(1+x)$




a)

✓ for correct $y''$.

✓ for test.

✓ for $(2,-27)$.

(b)

(i) ✓ for $y=1$ or $(0,1)$

(ii) ✓ for $-1$.

(iii)

✓ $x = \pm 1$

✓ $y = -1.$

(iv)

✓ for correct $y'$ (quotient rule)

✓ for $(0,1)$

✓ for testing.

(v)

✓✓ for correct shape with asymptotes

Solution of Ext1 Task 3 - 2010

Marks

(b)

(i) $6x + y = 60.$

(i) ✓

(ii) $A = xy + x(y+2x)$

$= xy + xy + 2x^2$

$= 2xy + 2x^2.$

$= 2x(60-6x) + 2x^2$

$= 120x - 12x^2 + 2x^2.$

$= 120x - 10x^2.$

(ii)

✓ for correct st $A = 2xy + 2x^2$

✓ for $120x - 10x$

(iii) $\frac{dA}{dx} = 120 - 20x = 0.$

$120 = 20x.$

$\underline{x = 6}$ cm.

$y = 60 - 6x = 60 - 36 = \underline{24cm}.$

$\frac{d^2A}{dx^2} = -20 < 0 \quad \therefore$ Max.

(iii)

✓ for correct $x$

✓ for $y = 24$ cm <u>and</u> testing (both

Follow through errors in iii. If area is wrong they lose the 2 marks in (ii)