MATHEMATICS (EXTENSION 1)

2014 Preliminary Course Assessment Task 2

Wednesday April 9, 2014

General instructions

- Working time – 75 minutes.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.
- Commence each new question on a new page. Write on both sides of the paper.
- **All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.**

Class (please ✔)

- ◯ 11M3A – Ms Ziaziaris
- ◯ 11M3B – Mr Ireland
- ◯ 11M3C – Mr Lin
- ◯ 11M3D – Mr Weiss
- ◯ 11M3E – Mr Berry
- ◯ 11M3F – Mr Lam

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Marker's use only.

QUESTION	1	2	3	4	5	6	**Total**	%
MARKS	$\overline{15}$	$\overline{15}$	$\overline{11}$	$\overline{12}$	$\overline{15}$	$\overline{14}$	$\overline{82}$	

Question 1 (15 Marks) Commence a NEW page. **Marks**

(a) $f(x)$ is defined as

$$f(x) = \begin{cases} x^2 & x < 2 \\ 6 - x & x > 2 \end{cases}$$

i. Evaluate $f(-1)$ and $f(6)$. **2**

ii. Sketch the graph of $y = f(x)$ within the domain $-1 \le x \le 6$. **3**

(b) If $f(x) = 5x - 7$, evaluate $f\left(\dfrac{7-x}{5}\right)$. **2**

(c) Show that $f(x) = \dfrac{1}{x\left(x^2-1\right)}$ is an odd function. **2**

(d) Sketch the following graphs, showing all important features:

i. $(x-2)^2 + (y+1)^2 = 4$. **3**

ii. $y = \dfrac{x-1}{x+4}$. **3**

Question 2 (15 Marks) Commence a NEW page. **Marks**

(a) Fully factorise:

i. $10x^2 + 11x - 6$. **1**

ii. $x^3 - 9x$. **1**

iii. $x^3 + x^2 + x + 1$. **2**

(b) Find a value for x such that $\sqrt{18} - \sqrt{2} = \sqrt{x}$. **2**

(c) Fully simplify: $\sqrt[3]{\dfrac{x^{5k+1}y^{4k-5}}{x^{2(k-1)}y^{-2k+1}}}$ **2**

(d) Solve the following for x:

i. $(3x+1)^2 = 6x + 2$. **2**

ii. $|3x - 5| = 5x$. **2**

iii. $\dfrac{5}{x-1} \ge 2$ **3**

Question 3 (11 Marks) Commence a NEW page. **Marks**

(a) Write 36° in exact form in radians. **1**

(b) State the exact value of $\cot \dfrac{11\pi}{6}$. **2**

(c) Fully simplify:

i. $\cos(x+y)\cos y + \sin(x+y)\sin y$. **1**

ii. $\cos^4\theta - \sin^4\theta$. **2**

(d) Prove, showing full working: **3**

$$\frac{\cos\theta}{1-\sin\theta} + \frac{1-\sin\theta}{\cos\theta} = 2\sec\theta$$

(e) Given $\tan 22.5^\circ = \sqrt{2}-1$, express $\tan 67.5^\circ$ in the form $a+\sqrt{b}$. **2**

Question 4 (12 Marks) Commence a NEW page. **Marks**

(a) Given that $\sec\theta = \dfrac{13}{5}$ and $\tan\theta < 0$, find the exact value of the following:

i. $\sin\theta$. **2**

ii. $\cos 2\theta$. **2**

(b) Solve for θ:

i. $\cos\theta = -\dfrac{\sqrt{3}}{2}$ for $-180^\circ \le \theta \le 180^\circ$. **2**

ii. $\sqrt{2}\sin 3x = -1$ for $0^\circ \le \theta \le 360^\circ$ **3**

iii. $2\sin^2\theta + 3\cos\theta = 3$ for $0^\circ \le \theta \le 360^\circ$ **3**

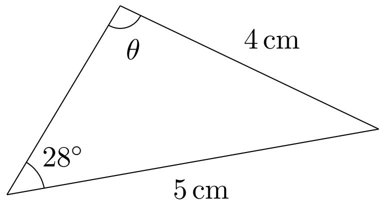
Question 5 (15 Marks) Commence a NEW page. **Marks**

(a) For the function $f(x) = -\dfrac{5\cos 3x}{2}$,

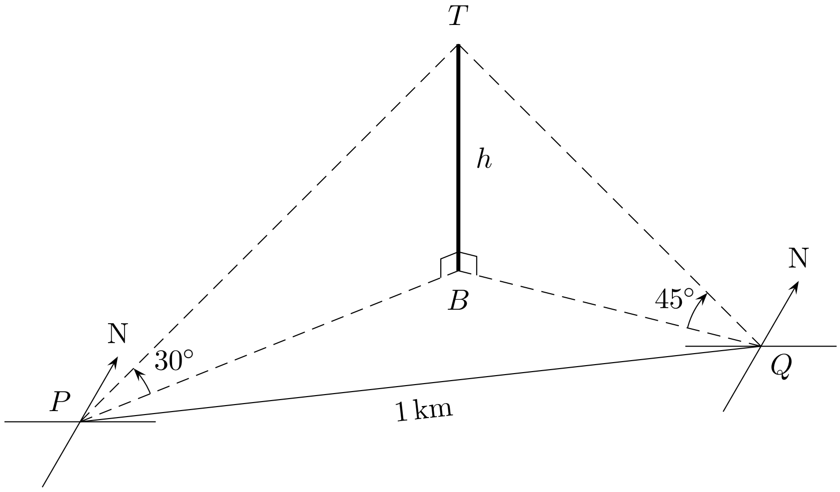
i. State the period (in degrees) and amplitude. **2**

ii. Sketch the graph of $y = f(x)$ for $0^\circ \leq x \leq 360^\circ$. **3**

(b) Find the value(s) of θ: **2**



(c) The angle of elevation from a boat P to a point T at the top of a vertical cliff is 30°. The boat sails 1 km to a second point Q, from which the angle of elevation to T is 45°. Let B be the point at the base of the cliff directly below T, and let $h = BT$ be the height of the cliff in metres.



The bearings of B from P and Q are 050°T and 310°T respectively. Copy or trace the diagram into your working booklet.

i. Show that $\angle PBQ = 100^\circ$. **2**

ii. Find expressions for PB and QB in terms of h. **1**

iii. Hence show that **2**

$$h^2 = \frac{1\,000^2}{4 - 2\sqrt{3}\cos 100^\circ}$$

iv. Hence or otherwise, calculate the height of the cliff correct to the nearest metre. **1**

v. Find the area of the sea that $\triangle PBQ$ encloses. **2**

Question 6 (14 Marks) Commence a NEW page. **Marks**

(a) Draw a neat sketch of $y = |4 - x| - x$. **3**

(b) Solve simultaneously: **3**

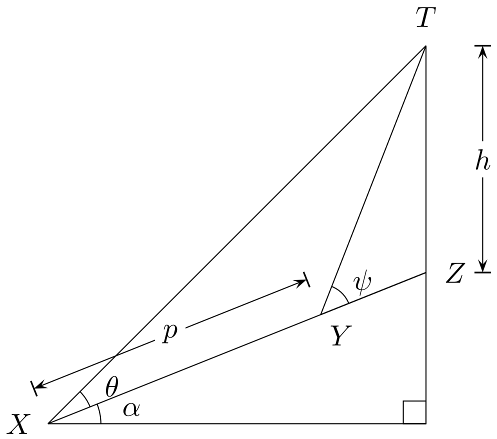
$$\begin{cases} y = 1 - \dfrac{3}{x+2} \\ y = 2x \end{cases}$$

(c) Solve for $0 \le x \le 2\pi$, giving your solutions as exact values: **3**

$$\sin x - \sin 2x = 0$$

(d) A ramp XYZ, rises at an angle of α with $XY = p$ metres. At Z there is a building of height h metres.

The angles of elevation of T from X and Y are $(\theta + \alpha)$ and $(\psi + \alpha)$ respectively.



i. Show that $TY = \dfrac{p \sin \theta}{\sin(\psi - \theta)}$ **1**

ii. Hence show that **2**

$$h = \frac{p \sin \theta \sin \psi}{\sin(\psi - \theta) \cos \alpha}$$

iii. Given that $\alpha = \theta = \dfrac{\pi}{6}$ and $\psi = \dfrac{\pi}{4}$, find the exact value of h in terms of p. **2**

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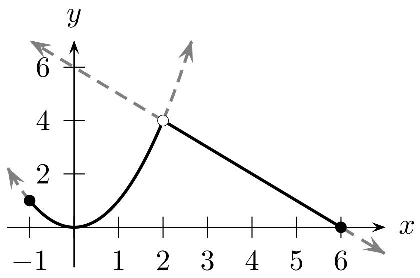
Suggested Solutions

Question 1 (Ireland)

(a) i. (2 marks)

$$f(x)=\begin{cases}x^2 & x<2\\ 6-x & x>2\end{cases}$$
$$f(1)=(-1)^2=1$$
$$f(6)=6-6=0$$

ii. (3 marks)



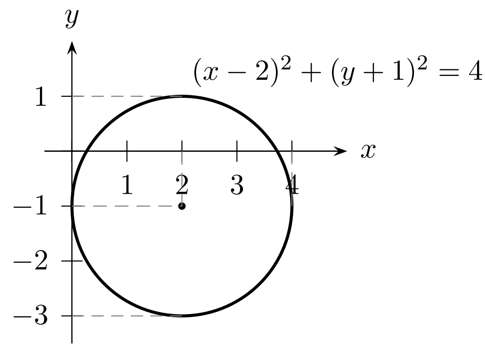
(b) (2 marks)

$$\begin{aligned}f(x)&=5x-7\\ f\left(\frac{7-x}{5}\right)&=\cancel{5}\left(\frac{7-x}{\cancel{5}}\right)-7\\ &=7-x-7\\ &=-x\end{aligned}$$

(c) (2 marks)

$$\begin{aligned}f(x)&=\frac{1}{x\left(x^2-1\right)}\\ f(-x)&=\frac{1}{(-x)\left((-x)^2-1\right)}\\ &=-\frac{1}{x\left(x^2-1\right)}\\ &=-f(x)\\ f(-x)&=-f(x)\\ \therefore f(x) &\text{ is odd}\end{aligned}$$

(d) i. (3 marks)



ii. (3 marks)

$$\begin{aligned}y&=\frac{x-1}{x+4}\\ &=\frac{x+4-5}{x+4}\\ &=1-\frac{5}{x+4}\end{aligned}$$

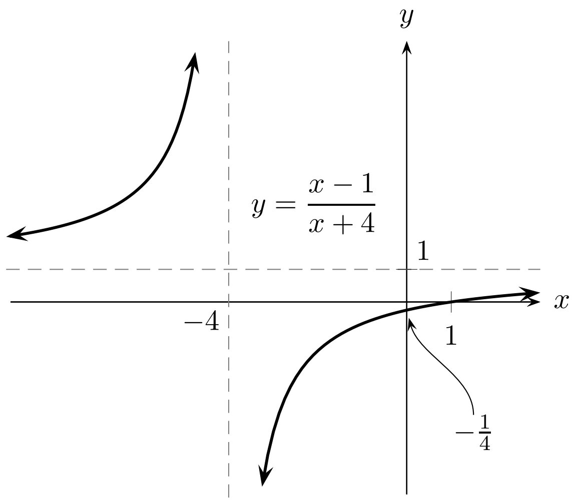
Intercepts:

- $x=0$:

$$y=1-\frac{5}{0+4}=-\frac{1}{4}$$

- $y=0$:

$$\begin{aligned}0&=1-\frac{5}{x+4}\\ \frac{5}{x+4}&=1\\ x+4&=5\\ x&=1\end{aligned}$$



Question 2 (Ireland)

(a) i. (1 mark)

$$10x^2 + 11x - 6 = (5x-2)(2x+3)$$

ii. (1 mark)

$$\begin{aligned} x^3 - 9x &= x(x^2-9) \\ &= x(x-3)(x+3) \end{aligned}$$

iii. (2 marks)

$$\begin{aligned} & x^3 + x^2 + x + 1 \\ &= x^2(x+1) + (x+1) \\ &= (x+1)\left(x^2+1\right) \end{aligned}$$

(b) (2 marks)

$$\begin{aligned} \sqrt{18} - \sqrt{2} &= 3\sqrt{2} - \sqrt{2} \\ &= 2\sqrt{2} \\ &= \sqrt{8} \\ \therefore x &= 8 \end{aligned}$$

(c) (2 marks)

$$\begin{aligned} & \sqrt[3]{\frac{x^{5k+1}y^{4k-5}}{x^{2(k-1)}y^{-2k+1}}} \\ &= \sqrt[3]{x^{5k+1-2(k-1)}y^{4k-5-(-2k+1)}} \\ &= \sqrt[3]{x^{3k+3}y^{6k-6}} \\ &= x^{k+1}y^{2(k-1)} \end{aligned}$$

(d) i. (2 marks)

$$\begin{aligned} (3x+1)^2 &= 6x+2 \\ 9x^2 + \cancel{6x} + 1 &= \cancel{6x} + 2 \\ 9x^2 &= 1 \\ x^2 &= \frac{1}{9} \\ x &= \pm\frac{1}{3} \end{aligned}$$

ii. (2 marks)

$$|3x-5| = 5x$$

When $(3x-5) \geq 0$,

$$\begin{aligned} 3x - 5 &= 5x \\ 2x &= -5 \\ x &= -\frac{5}{2} \end{aligned}$$

When $(3x-5) < 0$,

$$\begin{aligned} -(3x-5) &= 5x \\ -3x + 5 &= 5x \\ 8x &= 5 \\ x &= \frac{5}{8} \end{aligned}$$

Test values:

- $x = -\frac{5}{2}$, $5x$ becomes negative, whilst the left is positive. Hence invalid solution.
- $x = \frac{5}{8}$:

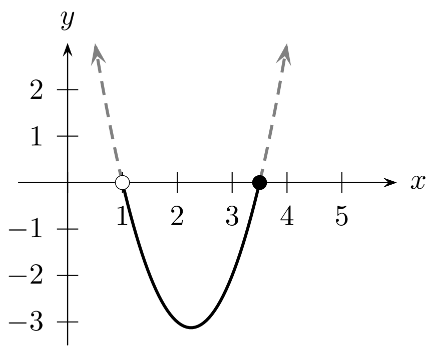
$$\begin{aligned} \left|3\left(\frac{5}{8}\right) - 5\right| &= \left|\frac{15}{8} - \frac{40}{8}\right| \\ &= \left|-\frac{25}{8}\right| \\ &= \frac{25}{8} \\ 5\left(\frac{5}{8}\right) &= \frac{25}{8} \end{aligned}$$

Hence $x = \frac{5}{8}$ is the only solution.

(e) (3 marks)

$$\underset{\times(x-1)^2}{\frac{5}{x-1}} \geq \underset{\times(x-1)^2}{2} \qquad x \neq 1$$

$$\begin{aligned} 5(x-1) &\geq 2(x-1)^2 \\ 2(x-1)^2 - 5(x-1) &\leq 0 \\ (x-1)\left(2(x-1)-5\right) &\leq 0 \\ (x-1)(2x-7) &\leq 0 \end{aligned}$$



$$\therefore 1 < x \leq \frac{7}{2}$$

Question 3 (Ziaziaris)

(a) (1 mark)

$$\begin{aligned} \underset{\div 5}{180^\circ} &\leftrightarrow \underset{\div 5}{\pi} \\ \therefore 36^\circ &\leftrightarrow \frac{\pi}{5} \end{aligned}$$

(b) (2 marks)

$$\begin{aligned} \cot\frac{11\pi}{6} &= \cot\left(2\pi - \frac{\pi}{6}\right) \\ &= -\cot\frac{\pi}{6} \\ &= -\frac{1}{\frac{1}{\sqrt{3}}} \\ &= -\sqrt{3} \end{aligned}$$

(c) i. (1 mark)

$$\begin{aligned} &\cos(x+y)\cos y + \sin(x+y)\sin y \\ &= \cos[(x+y) - y] \\ &= \cos x \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} &\cos^4\theta - \sin^4\theta \\ &= \cancel{\left(\cos^2\theta + \sin^2\theta\right)}\underbrace{\left(\cos^2\theta - \sin^2\theta\right)}_{=\cos 2\theta} \\ &= \cos 2\theta \end{aligned}$$

(d) (3 marks)

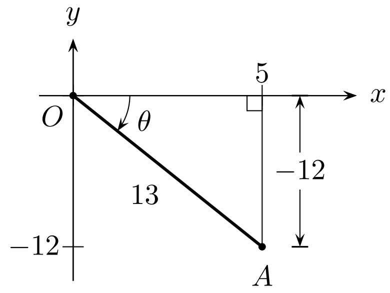
$$\begin{aligned} &\frac{\cos\theta}{1-\sin\theta}{\scriptstyle\frac{\times\cos\theta}{\times\cos\theta}} + \frac{1-\sin\theta}{\cos\theta}{\scriptstyle\frac{\times(1-\sin\theta)}{\times(1-\sin\theta)}} \\ &= \frac{\cos^2\theta + (1-\sin\theta)^2}{\cos\theta(1-\sin\theta)} \\ &= \frac{\cos^2\theta + 1 - 2\sin\theta + \sin^2\theta}{\cos\theta(1-\sin\theta)} \\ &= \frac{2 - 2\sin\theta}{\cos\theta(1-\sin\theta)} \\ &= \frac{2\cancel{(1-\sin\theta)}}{\cos\theta\cancel{(1-\sin\theta)}} \\ &= 2\sec\theta \end{aligned}$$

(e) (2 marks)

$$\begin{aligned} \tan 22.5^\circ &= \sqrt{2} - 1 \\ \tan 67.5^\circ &= \tan(22.5^\circ + 45^\circ) \\ &= \frac{\tan 22.5^\circ + \tan 45^\circ}{1 - \tan 22.5^\circ \tan 45^\circ} \\ &= \frac{\left(\sqrt{2}-1\right)+1}{1-\left(\sqrt{2}-1\right)} \\ &= \frac{\sqrt{2}}{2-\sqrt{2}}{\scriptstyle\frac{\times 2+\sqrt{2}}{\times 2+\sqrt{2}}} \\ &= \frac{2\sqrt{2}+2}{4-2} \\ &= 1 + \sqrt{2} \end{aligned}$$

Question 4 (Lam)

(a) i. (2 marks)



$$\sin\theta = -\frac{12}{13}$$

ii. (2 marks)

$$\begin{aligned} \cos\theta &= \frac{5}{13} \\ \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{5}{13}\right)^2 - 1 \\ &= 2 \times \frac{25}{169} - 1 \\ &= \frac{50}{169} - \frac{169}{169} \\ &= -\frac{119}{169} \end{aligned}$$

(b) i. (2 marks)

$$\begin{aligned} \cos\theta &= -\frac{\sqrt{3}}{2} \\ \theta &= \pm 150^\circ \end{aligned}$$

ii. (3 marks)

$$\sqrt{2}\sin 3x = -1 \quad 0^\circ \le x \le 360^\circ$$
$$\sin 3x = -\frac{1}{\sqrt{2}} \quad 0^\circ \le 3x \le 1080^\circ$$
$$\underset{\div 3}{3x} = \underbrace{225^\circ, 315^\circ, 585^\circ, 675^\circ, 945^\circ, 1035^\circ}_{\div 3}$$
$$x = 75^\circ, 105^\circ, 195^\circ, 225^\circ, 315^\circ, 345^\circ$$

iii. (3 marks)

$$2\sin^2\theta + 3\cos\theta = 3$$
$$2\left(1 - \cos^2\theta\right) + 3\cos\theta = 3$$
$$2 - 2\cos^2\theta + 3\cos\theta = 3$$
$$2\cos^2\theta - 3\cos\theta + 1 = 0$$

Let $m = \cos\theta$,

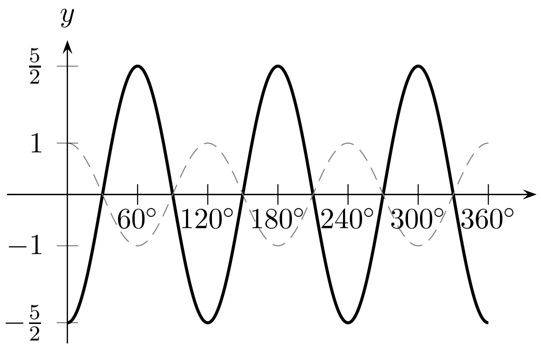
$$2m^2 - 3m + 1 = 0$$
$$(2m-1)(m-1) = 0$$
$$m = 1, \frac{1}{2}$$
$$\therefore \cos\theta = 1, \frac{1}{2}$$
$$\theta = 0^\circ, 60^\circ, 300^\circ, 360^\circ$$

Question 5 (Lin)

(a) i. (2 marks)

- $T = \frac{360^\circ}{3} = 120^\circ$
- $a = \frac{5}{2}$

ii. (3 marks)

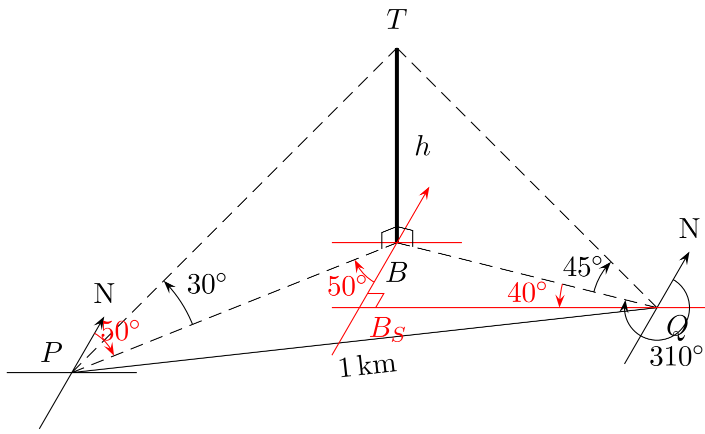


(b) (2 marks)

$$\frac{\sin\theta}{5} = \frac{\sin 28^\circ}{4}$$
$$\sin\theta = \frac{5\sin 28^\circ}{4}$$
$$\theta = 35^\circ 56', 144^\circ 4'$$

(Ambiguous case)

(c) i. (2 marks)



- $\angle PBB_S = 50^\circ$
  (alternate $\angle$, $NP \parallel BB_S$)
- $\angle BQB_S = 310^\circ - 270^\circ = 40^\circ$
- $\angle BB_SQ = 90^\circ$
- Hence

$$\angle QBB_S = 90^\circ - 40^\circ = 50^\circ$$

(complementary)

- Hence $\angle PBQ = 50^\circ + 50^\circ = 100^\circ$

ii. (1 mark)

$$\frac{h}{PB} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$
$$\therefore PB = h\sqrt{3}$$

In $\triangle TBQ$,

$$\frac{h}{QB} = \tan 45^\circ = 1$$
$$\therefore QB = h$$

iii. (2 marks)
Applying the cosine rule in $\triangle PBQ$, where $PQ = 1\,000$ m,

$$1\,000^2 = PB^2 + QB^2 - 2(PB)(QB)\cos 100^\circ$$
$$1\,000^2 = \left(h\sqrt{3}\right)^2 + h^2 - 2\left(h\sqrt{3}\right)(h)\cos 100^\circ$$

Factorising h^2,

$$1\,000^2 = h^2\left(3 + 1 - 2\sqrt{3}\cos 100^\circ\right)$$
$$\therefore h^2 = \frac{1\,000^2}{4 - 2\sqrt{3}\cos 100^\circ}$$

iv. (1 mark)

$$h = 466\,\mathrm{m}$$

v. (2 marks)

$$\begin{aligned} PB &= h\sqrt{3} \quad QB = h \\ A &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2} \times h\sqrt{3} \times h \times \sin 100^\circ \\ &= \frac{1}{2} \times \frac{1\,000^2 \times \sqrt{3}\sin 100^\circ}{4 - 2\sqrt{3}\cos 100^\circ} \\ &= 185\,344\,\mathrm{m}^2 \end{aligned}$$

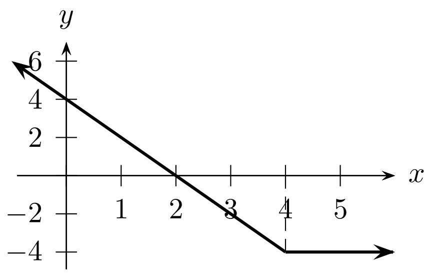
Question 6 (Zuber)

(a) (3 marks)

$$y = |-(x-4)| - x$$

Examine $|-(x-4)|$:

$$\begin{aligned} |-(x-4)| &= |x-4| \\ &= \begin{cases} (x-4) & (x-4) \ge 0 \\ -(x-4) & (x-4) < 0 \end{cases} \\ &= \begin{cases} (x-4) & x \ge 4 \\ -(x-4) & x < 4 \end{cases} \\ \therefore y &= \begin{cases} (x-4) - x & x \ge 4 \\ -(x-4) - x & x < 4 \end{cases} \\ &= \begin{cases} -4 & x \ge 4 \\ -2x + 4 & x < 4 \end{cases} \end{aligned}$$



(b) (3 marks)

$$\begin{cases} y = 1 - \dfrac{3}{x+2} \\ y = 2x \end{cases}$$

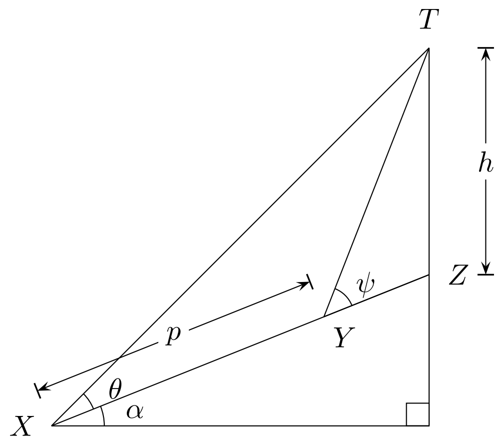
Equating,

$$\begin{aligned} 2x &= 1 - \frac{3}{x+2} \\ 2x(x+2) &= (x+2) - 3 \\ 2x^2 + 4x &= x - 1 \\ 2x^2 + 3x + 1 &= 0 \\ (2x+1)(x+1) &= 0 \\ x &= -\frac{1}{2}, -1 \\ y &= -1, -2 \end{aligned}$$

(c) (3 marks)

$$\begin{aligned} \sin x - \sin 2x &= 0 \\ \sin x - 2\sin x\cos x &= 0 \\ \sin x\,(1 - 2\cos x) &= 0 \\ \sin x = 0 \quad \cos x &= \frac{1}{2} \\ \therefore x = 0, \pi, 2\pi, \quad & \frac{\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

(d) i. (1 mark)



In $\triangle TYX$, $\angle XTY = \psi - \theta$. Applying the sine rule,

$$\begin{aligned} \frac{TY}{\sin\theta} &= \frac{p}{\sin(\psi - \theta)} \\ \therefore TY &= \frac{p\sin\theta}{\sin(\psi - \theta)} \end{aligned}$$

ii. (2 marks)
In $\triangle TYZ$,

$$\begin{aligned}
\angle TZY &= \pi - \left(\frac{\pi}{2} - \alpha\right) \\
&= \frac{\pi}{2} + \alpha \\
\frac{h}{\sin\psi} &= \frac{TY}{\sin\left(\frac{\pi}{2}+\alpha\right)} \\
\therefore h &= \frac{TY\sin\psi}{\sin\left(\frac{\pi}{2}+\alpha\right)} \\
&= \frac{p\sin\theta\sin\psi}{\sin(\psi-\theta)\sin\left(\frac{\pi}{2}+\alpha\right)} \\
&= \frac{p\sin\theta\sin\psi}{\sin(\psi-\theta)\left(\sin\frac{\pi}{2}\cos\alpha + \cancel{\cos\frac{\pi}{2}\sin\alpha}\right)} \\
&= \frac{p\sin\theta\sin\psi}{\sin(\psi-\theta)\cos\alpha}
\end{aligned}$$

iii. (2 marks)

$$\alpha = \theta = \frac{\pi}{6}, \psi = \frac{\pi}{4}$$

$$\begin{aligned}
h &= \frac{p\sin\frac{\pi}{6}\sin\frac{\pi}{4}}{\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\cos\frac{\pi}{6}} \\
&= \frac{p\times\frac{1}{2}\times\frac{1}{\sqrt{2}}}{\left(\sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}} \\
&= \frac{\frac{p}{2\sqrt{2}}}{\frac{\sqrt{3}}{2}\times\left(\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2}\right)} \\
&= \frac{\frac{p}{2\sqrt{2}}}{\frac{3}{4\sqrt{2}} - \frac{\sqrt{3}}{4\sqrt{2}}} \\
&= \frac{2p}{3-\sqrt{3}}\,{}^{\times 3+\sqrt{3}}_{\times 3+\sqrt{3}} \\
&= \frac{2p\left(3+\sqrt{3}\right)}{9-3} \\
&= \frac{p\left(3+\sqrt{3}\right)}{3}
\end{aligned}$$