North Sydney Boys High School

MATHEMATICS (EXTENSION 1)

2015 HSC Course Assessment Task 4

Friday August 14, 2015

General instructions

- Working time – 55 min. (plus 5 minutes reading time)
- **Commence each new question on a new page.**
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M3A – Ms Ziaziaris
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Zuber
- ◯ 12M4A – Mr Weiss
- ◯ 12M4B – Mr Lin
- ◯ 12M4C – Mr Ireland

STUDENT NUMBER .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	4	**Total**	%
MARKS	$\overline{}$ 8	$\overline{}$ 12	$\overline{}$ 9	$\overline{}$ 8	$\overline{}$ 37	

Question 1 (8 Marks) Commence a NEW page. **Marks**

(a) Find the constant term in the expression of $\left(x - \dfrac{1}{2x^3}\right)^{20}$. **3**

(b) Prove that $\dbinom{n+1}{r} = \dbinom{n}{r-1} + \dbinom{n}{r}$. **2**

(c) In the expression of $(2+3x)^n$, the coefficients of x^3 and x^4 are in the ratio $8 : 15$. **3**

Find the value of n.

Question 2 (12 Marks) Commence a NEW page. **Marks**

(a) A cat is sitting on the top of a wall measuring 3.2 m high. It sees a mouse on the ground 4 m away from the foot of the wall. The cat jumps horizontally from the top of the wall at an initial speed of $6\,\mathrm{ms}^{-1}$. Take $g = 10\,\mathrm{ms}^{-2}$.

i. Show that the equations of motion are **2**

$$\begin{cases} x = 6t \\ y = -5t^2 + 3.2 \end{cases}$$

ii. How long does it take the cat to reach the ground? **1**

iii. Show that the cat misses the mouse. **1**

iv. If the cat is to land on the mouse, calculate the speed at which the cat should leap off the wall. **1**

(b) A projectile is fired from a point P on horizontal ground with initial speed V ms^{-1} at an angle of $45°$. The only force acting on the projectile is due to gravity, measuring g ms^{-2}.

i. State the equations of motion. **1**

ii. Determine the coordinates (X, Y) of the point at its greatest height. **2**

iii. Prove that a second projectile, fired from P with an initial speed V ms^{-1}, at an angle of elevation of θ, where $\tan\theta = 3$, will also pass through (X, Y). **2**

iv. If the projectiles are fired simultaneously, what are the coordinates of the second projectile at the instant the first projectile lands? **2**

Question 3 (9 Marks) Commence a NEW page. **Marks**

(a) A particle moves along the x axis. Its velocity is given by

$$v = \pm\sqrt{8x - x^2}$$

i. Prove the particle is undergoing simple harmonic motion. **2**

ii. Find the acceleration of the particle when $x = 3$. **1**

iii. Find the amplitude of the motion. **2**

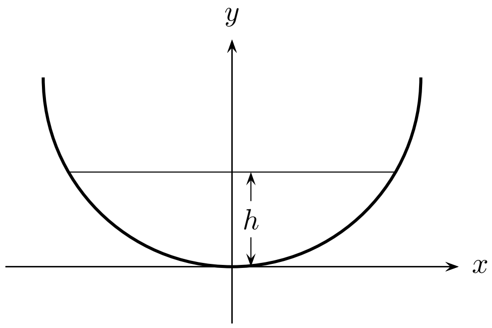
(b) Assuming that tides rise and fall in simple harmonic motion, a ship needs 11 m of water to pass through a channel safely. At low tide, the channel is 8 m deep and at high tide it is 12 m. Low tide occurs at 10 am and high tide occurs at 4 pm. **4**

Find the period of time for which the ship can safely proceed through the channel.

Question 4 (8 Marks) Commence a NEW page. **Marks**

The diagram shows the lower half of the circle whose equation is

$$x^2 + y^2 - 20y = 0$$



A hemispherical bowl is obtained by rotating the semicircle about the y axis.

(a) Show that when the depth of water is h cm:

i. The volume of water V is given by **2**

$$V = \frac{\pi h^2}{3}(30 - h)\ \text{cm}^3$$

ii. The surface area of water exposed to the air is given by **2**

$$S = \pi h(20 - h)\ \text{cm}^2$$

(b) If water is poured in at $50\,\text{cm}^3/\text{s}$, determine the following when the depth of water is 5 cm:

i. the rate of increase of the depth of water. **2**

ii. the rate of increase of the surface area of the water. **2**

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \qquad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right) + C$$

NOTE: $\ln x = \log_e x,\ x > 0$

Q1.

(a) $$\left(x - \frac{1}{2x^3}\right)^{20} = \sum_{k=0}^{20} \binom{20}{k} x^{20-k} \left(-\frac{1}{2}\right)^k x^{-3k}$$

$$= \sum_{k=0}^{20} \binom{20}{k} x^{20-4k} \left(-\frac{1}{2}\right)^k$$

x^0 term when $20-4k=0$
ie $k=5$ ✓

Term is: $\binom{20}{5}\left(-\frac{1}{2}\right)^5 = -\frac{969}{2}$ ✓

(b) $${}^nC_r + {}^nC_{\cancel{r+1}} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!(n-r+1)}{r!(n-r+1)!} + \frac{r \cdot n!}{r!(n-r+1)!}$$ ✓

$$= \frac{n!(n-r+1+r)}{r!(n-r+1)!}$$

$$= \frac{(n+\cancel{r})!}{r!(n-r+1)!}$$ ✓

$$= {}^{n+1}C_r$$

<u>or</u>: use coefficients of $(1+x)^{n+1} = (1+x)(1+x)^n$

LHS coeff on x^r is ${}^{n+1}C_r$

RHS coeff of x^r is $({}^nC_{r-1})x^{r-1} \times x + 1 \cdot {}^nC_r x^r$

$$= {}^nC_{r-1} + {}^nC_r$$

$$\therefore {}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$$

Q1 (c) Ratio coefficients:

$$\frac{{}^nC_3\, 2^{n-3}\, 3^3}{{}^nC_4\, 2^{n-4}\, 3^4} = \frac{8}{15}$$

$\checkmark 1$

$$\frac{{}^nC_3}{{}^nC_4} \cdot \frac{2}{3} = \frac{8}{15}$$

$$\frac{{}^nC_3}{{}^nC_4} = \frac{24}{30}$$

$$\frac{4}{n-3} = \frac{4}{5}$$

$\therefore \quad n = 8$

Question 2

(a)(i) $\ddot{x} = 0$

$\dot{x} = c$

$\dot{x} = 6$ (horizontal)

$x = \int 6\,dt$

$x = 6t + c$

when $t = 0$ $x = 0$

$\therefore x = 6t$

$\ddot{y} = -10$

$\dot{y} = \int -10\,dt$

$\dot{y} = -10t + c$

when $t = 0$ $\dot{y} = 0$

$\therefore 0 = -10(0) + c$

$c = 0$

$\dot{y} = -10t$

$y = \int -10t\,dt$

$y = -5t^2 + c$

when $t = 0$ $y = 3.2$

$3.2 = -5(0) + c$

$\therefore c = 3.2$

$\therefore y = -5t^2 + 3.2$

(ii) when $y = 0$

$0 = -5t^2 + 3.2$

$5t^2 = 3.2$

$t^2 = 3.2/5$

$t = \pm\sqrt{3.2/5}$

$\therefore t = 0.8$ $(t > 0)$

(iii) when $t = 0.8$

$x = 6(0.8)$

$x = 4.8$ m

$\therefore$ the cat lands at $x = 4.8$ and misses the mouse which is at $x = 4$ m.

(iv) speed $= \dfrac{\text{distance}}{\text{time}}$

$= 4 \div (0.8)$

$= 5$ m/s

(b)(i) $\ddot{x} = 0$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta$

$\ddot{y} = -g$

$\dot{y} = -gt + V\sin\theta$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$

since $\sin 45° = \frac{1}{\sqrt{2}}$ and $\cos 45° = \frac{1}{\sqrt{2}}$ ..

$\ddot{x} = 0$

$\dot{x} = V/\sqrt{2}$

$x = Vt/\sqrt{2}$

$\ddot{y} = -g$

$\dot{y} = -gt + V/\sqrt{2}$

$y = -\frac{1}{2}gt^2 + Vt/\sqrt{2}$

(ii) when $\dot{y} = 0$

$-gt + V/\sqrt{2} = 0$

$gt = V/\sqrt{2}$

$t = \frac{V}{g\sqrt{2}}$ ... ①

sub ① into x & y

$x = Vt/\sqrt{2}$

$x = \frac{V\left(\frac{V}{g\sqrt{2}}\right)}{\sqrt{2}}$

$x = V^2/2g$

$y = -\frac{1}{2}gt^2 + Vt/\sqrt{2}$

$y = -\frac{1}{2}g\left(\frac{V}{g\sqrt{2}}\right)^2 + \frac{V\left(\frac{V}{g\sqrt{2}}\right)}{\sqrt{2}}$

$y = -\frac{V^2\cancel{g}}{4g^{\cancel{2}}} + \frac{V^2}{2g}$

$y = \frac{V^2}{4g}$

∴ coordinates are

$(V^2/2g\,,\ V^2/4g)$

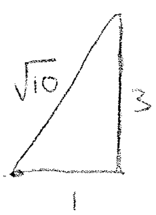
(iii) equations of motion are:

$\ddot{x} = 0$ $\quad \ddot{y} = -g$

$\dot{x} = V\cos\theta$ $\quad \dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta$ $\quad y = -\frac{1}{2}gt^2 + Vt\sin\theta$

since $\tan\theta = 3$



$\sin\theta = \frac{3}{\sqrt{10}}$

$\cos\theta = \frac{1}{\sqrt{10}}$

∴ equations of motion are

$\ddot{x} = 0$

$\dot{x} = V/\sqrt{10}$

$x = Vt/\sqrt{10}$

$\ddot{y} = -g$

$\dot{y} = -gt + 3V/\sqrt{10}$

$y = -\frac{1}{2}gt^2 + 3Vt/\sqrt{10}$

find when $x = \frac{V^2}{2g}$

$\cancel{V}t/\sqrt{10} = V^{\cancel{2}}/2g$

$t = V\sqrt{10}/2g$

show that when $t = \frac{V\sqrt{10}}{2g}$

that $y = V^2/4g$

$y = -\frac{1}{2}gt^2 + \frac{3Vt}{\sqrt{10}}$

$y = -\frac{1}{2}g\left(\frac{V\sqrt{10}}{2g}\right)^2 + \left(\frac{3V}{\sqrt{10}}\right)\left(\frac{V\sqrt{10}}{2g}\right)$

$= -\frac{10V^2\cancel{g}}{8g^{\cancel{2}}} + \frac{3V^2}{2g}$

$= -\frac{5V^2}{4g} + \frac{6V^2}{4g}$

$V^2/4g$, as required

∴ 2nd projectile passes through (X, Y) since when $x = V^2/2g$ $\quad y = \frac{V^2}{4g}$

(iii) <u>alternative solution</u>

$x = Vt\cos\theta \quad \ldots (1)$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta \ldots (2)$

rearrange (1) to get

$t = \frac{x}{V\cos\theta}$

and sub into (2)

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$

$y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$

$y = \frac{-gx^2}{2V^2\cos^2\theta} + x\tan\theta$

$y = \frac{-gx^2}{2V^2}\sec^2\theta + x\tan\theta$

$y = -\frac{gx^2}{2V^2}(\tan^2\theta + 1) + x\tan\theta$

we now have to show that $x = V^2/2g$, $y = V^2/4g$ & $\tan\theta = 3$ satisfy the cartesian equation above.

$LHS = V^2/4g$

$RHS = -\frac{g}{2V^2}\left(\frac{V^2}{2g}\right)^2(3^2+1) + \frac{V^2}{2g}(3)$

$= -\frac{gV^4(10)}{48V^2g^2} + \frac{3V^2}{2g}$

$= -\frac{5V^2}{4g} + \frac{6V^2}{4g}$

$= V^2/4g$

$= LHS$

$\therefore$ the projectile passes through (X, Y)

(iv) find t when $y = 0$ for the first projectile.

$\therefore -\frac{1}{2}gt^2 + Vt/\sqrt{2} = 0$

$\therefore t\left(-\frac{1}{2}gt + V/\sqrt{2}\right) = 0$

$-\frac{1}{2}gt + V/\sqrt{2} = 0 \quad (t > 0)$

$\frac{1}{2}gt = V/\sqrt{2}$

$t = \frac{2V}{g\sqrt{2}}$

sub into x & y for the 2nd projectile.

$x = Vt/\sqrt{10}$

$x = V/\sqrt{10}\left(\frac{2V}{g\sqrt{2}}\right)$

$x = 2V^2/g\sqrt{20}$

$x = V^2/g\sqrt{5}$

$y = -\frac{1}{2}gt^2 + 3Vt/\sqrt{10}$

$y = -\frac{1}{2}g\left(2V/g\sqrt{2}\right)^2 + 3V/\sqrt{10}\left(2V/g\sqrt{2}\right)$

$y = -\frac{4V^2g}{2\times g^2\times 2} + \frac{6V^2}{g\sqrt{20}}$

$y = -V^2/g + 6V^2/2g\sqrt{5}$

$y = \frac{3V^2 - \sqrt{5}V^2}{g\sqrt{5}}$

$\therefore$ coordinates are

$\left(\frac{V^2}{g\sqrt{5}}, \frac{V^2(3-\sqrt{5})}{g\sqrt{5}}\right)$

a) (i) $v = \pm\sqrt{8x - x^2}$

$v^2 = 8x - x^2$

$\frac{1}{2}v^2 = 4x - \frac{x^2}{2}$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(4x - \frac{x^2}{2}\right)$

$\therefore \ddot{x} = 4 - x$

$\ddot{x} = -1(x - 4)$

which is in the form

$\ddot{x} = -n^2(x - x_0)$ where $n = 1$, x_0 = centre of motion is 4.

$\therefore$ Particle in SHM.

(OR you could say $\ddot{x}$ is proportional to the negative displacement)

(ii) $\ddot{x} = 4 - x$

When $x = 3$

$\ddot{x} = 1$

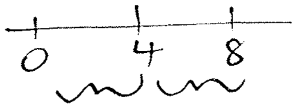
(iii) When $v = 0$.

$0 = 8x - x^2$

$x = 0,\ x = 8$.

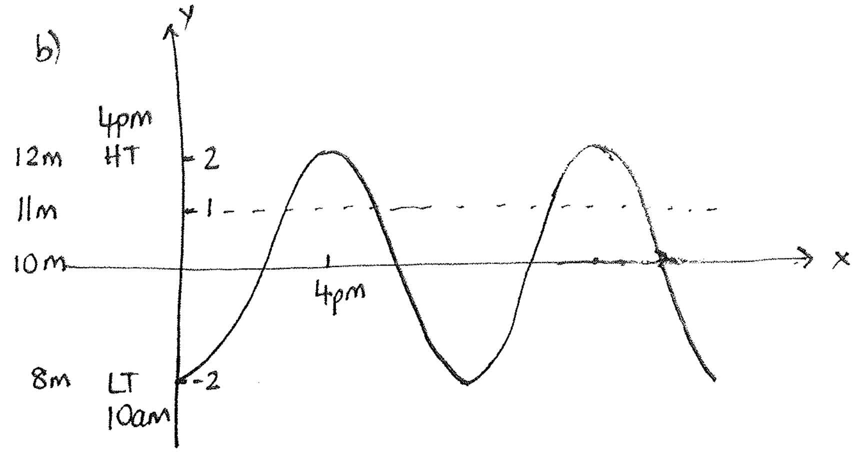
Comes to rest at $x = 0$ $x = 8$

Centre of motion : $x = 4$



$\therefore$ Amplitude $a = 4$

b)



Period = 12 hours

$\frac{2\pi}{n} = 12$

$\therefore n = \frac{\pi}{6}$ Amplitude = 2

$x = -a\cos nt$ $\xrightarrow{\text{OR}}$ $x = -a\cos nt + x_0$

$x = -2\cos\frac{\pi}{6}t$

$x = -2\cos\frac{\pi}{6}t + 10$

When $x = 11$

$11 = -2\cos\frac{\pi}{6}t + 10.$

When 11m, $x = 1$

$1 = -2\cos\frac{\pi}{6}t$

$\cos\frac{\pi}{6}t = -\frac{1}{2}$

$\frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$

$t = 4, 8$

Will be above 11m 4h and 8h after 10am

$\therefore$ Can enter through 2pm and 6pm.

BUT in 24h period can also enter

between 2am - 6am.

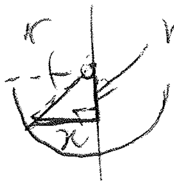
Q4. (a)(i) $V = \pi \int_0^h x^2 \, dy$

✓ $= \pi \int_0^h (20y - y^2) \, dy$

$= \pi \left[10y^2 - \frac{y^3}{3} \right]_0^h$

$= \pi \left(10h^2 - \frac{h^3}{3} \right)$

✓ $= \frac{\pi h^2}{3}(30 - h) \text{ cm}^3$

(ii) 

$x^2 = r^2 - (r-h)^2$

$r = 10$

$\therefore x^2 = 10^2 - (10 - h)^2$

$= 20h - h^2$

or $y = h \therefore x^2 + h^2 - 20h = 0$ ✓

$x^2 = 20h - h^2$

$S.A. = \pi x^2$

$= \pi(20h - h^2)$ ✓

$= \pi h(20 - h)$ as required.

(b)
(i) $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ (1)

Now: $\frac{dV}{dt} = 50$ (given)

$V = \pi \left(10h^2 - \frac{1}{3}h^3 \right)$

$\frac{dV}{dh} = \pi(20h - h^2)$

At $h = 5$,

$\frac{dV}{dh} = \pi(20(5) - 25)$

$= 75\pi$

$\therefore 50 = 75\pi \cdot \frac{dh}{dt}$ from (1)

ie $\frac{dh}{dt} = \frac{50}{75\pi}$

$= \frac{2}{3\pi}$ cm/sec

(ii) $\frac{dS}{dt} = \frac{dS}{dh} \cdot \frac{dh}{dt}$ (2)

$S = \pi(20h - h^2)$

$\frac{dS}{dh} = \pi(20 - 2h)$

At $h = 5$, $\frac{dS}{dh} = 10\pi$

$\therefore \frac{dS}{dt} = 10\pi \cdot \frac{2}{3\pi}$ (from (i))

$= \frac{20}{3}$ cm/s.