NORTH SYDNEY BOYS HIGH SCHOOL

2016 YEAR 12 ASSESSMENT TASK 1

Mathematics Extension 1

General Instructions

- **Working time – 60 minutes**
- **Reading time – 5 minutes**
- Answer all questions on the lined paper in the booklet provided.
- Write using blue or black pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Each new question is to be started on a **new page** with the exception of the multiple choice which can be answered on the same page.
- Attempt all questions.
- Use the reference sheet provided.

Class Teacher: Please tick

- ○ Mr Berry
- ○ Mr Ireland
- ○ Mr Jomaa
- ○ Ms Lee
- ○ Mr Lin
- ○ Ms Ziaziaris

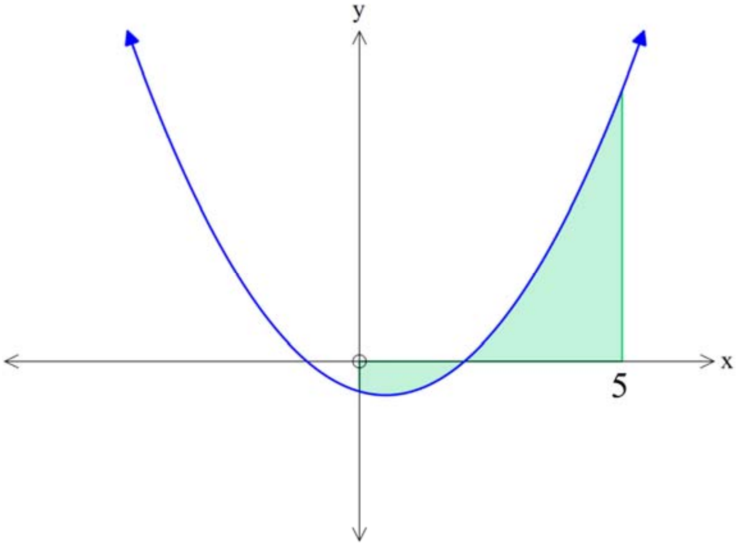
Student No./Name:___________________________

(To be used by the exam markers only.)

Section	A	B	C	D	E	F	Total	%
Mark	/3	/7	/9	/11	/8	/11	/49	/100

Section A (Start a new page)	**Marks**

1. Which expression will give the area of the shaded region bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 0$ and $x = 5$? **1**



(A) $$Area = \left|\int_0^1 (x^2 - x - 2)dx\right| + \int_1^5 (x^2 - x - 2)dx$$

(B) $$Area = \int_0^1 (x^2 - x - 2)dx + \left|\int_1^5 (x^2 - x - 2)dx\right|$$

(C) $$Area = \left|\int_0^2 (x^2 - x - 2)dx\right| + \int_2^5 (x^2 - x - 2)dx$$

(D) $$Area = \int_0^2 (x^2 - x - 2)dx + \left|\int_2^5 (x^2 - x - 2)dx\right|$$

2. Which of the following is an expression for $\int (2x+1)^{10} dx$ 1

(A) $\frac{1}{9}(2x+1)^9 + c$

(B) $\frac{1}{11}(2x+1)^{11} + c$

(C) $\frac{2}{11}(2x+1)^{11} + c$

(D) $\frac{1}{22}(2x+1)^{11} + c$

3. The equation of the normal to the parabola $x^2 = 4\alpha y$ the variable point $P(2\alpha p, \alpha p^2)$ is given by $x + py = 2\alpha p + \alpha p^3$. How many different values of p are there such that the normal passes through the focus of the parabola? 1

(A) 0
(B) 1
(C) 2
(D) 3

Section B (Start a new page)	**Marks**

1. Find $\displaystyle\int \left(x + \frac{1}{x}\right)^2 dx$ **2**

2. Find $\displaystyle\int \frac{5x^3 + 1}{x^2} dx$ **2**

3. Find $\displaystyle\int_1^4 \sqrt{2x}\, dx$ **3**

Section C (Start a new page)

1. In an arithmetic sequence with $T_9 = 3T_6$ and $T_4 + T_7 = -4$. Find the value of the first term. **3**

2. Find the sum of 10 terms of the series $\log_m 3 + \log_m 6 + \log_m 12 + \cdots$ given that $\log_m 3 = 0.48$ and $\log_m 2 = 0.30$ **2**

3. Evaluate $\displaystyle\sum_{r=1}^{11} [(-1)^r 3^r + (3r + 1)]$ **4**

Section D (Start a new page)

1. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ **3**

(ii) The equation of the chord PQ is $y = \frac{p+q}{2}x - apq$ (Do NOT show this.) If the chord PQ passes through the focus of the parabola, show that $pq = -1$ **2**

(iii) Write down the equation of the normal to the parabola at Q. The normals intersect at N.
Show that $N(-apq(p + q), a(p^2 + q^2 + pq + 2)$. **2**

(iv) If PQ is a focal chord and N is the intersection of the normals, find the equation of the locus of N and give a precise geometrical description. **4**

Section E (Start a new page)

1. When the polynomial $P(x)$ is divided by $x^2 - 5x + 6$, the remainder is $5x + 2$. What is the remainder when $P(x)$ is divided by $x - 2$? **2**

2. The cubic equation $2x^3 - x^2 + x + 1 = 0$ has roots α, β and γ. Evaluate **3**

(i) $\alpha\beta + \beta\gamma + \alpha\gamma$

(ii) $\alpha\beta\gamma$

(iii) $\alpha^2\beta^2\gamma + \beta^2\gamma^2\alpha + \alpha^2\gamma^2\beta$

3. Solve the polynomial equation $x^3 - 12x^2 + 12x + 80 = 0$ if the roots of this equation are in arithmetic progression. **3**

Section F (Start a new page)	**Marks**

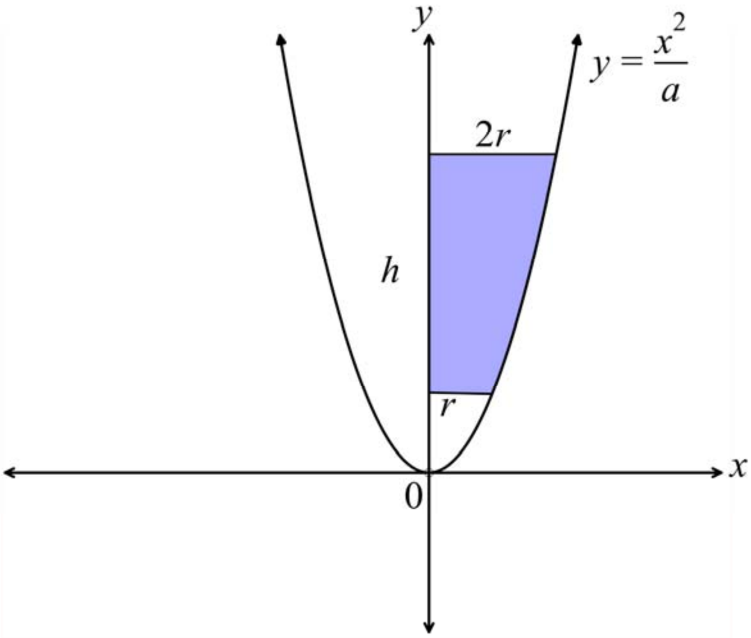
1. Use the trapezoidal rule with five function values, to find an approximate value of the area under the curve $y = \log_{10} x$, bounded by the x-axis and $x = 1$ and $x = 3$, correct to 4 significant figures. **2**

2. Consider the curves $x^2 = 4y$ and $y^2 = 4x$.

(i) Show that the curves intersect at (0, 0) and (4, 4) **1**

(ii) Find the area enclosed between the two curves. **3**

3. An open bowl with a flat circular base is made by rotating the shaded area of the curve $y = \frac{x^2}{a}$ around the y-axis. **5**



The radius of the top rim of the bowl is twice that of the base, i.e. $2r$ and r respectively.

The volume of the bowl is $\frac{5}{6}\pi a^3\ unit^3$.

Find the height (h) of the bowl.

End of examination

2016 HSC Task 1 Suggested Solutions

Section A

1. C
2. D
3. B

Section B

1. $\int (x+\frac{1}{x})^2 dx$ $\boxed{2}$

$= \int (x^2 + 2 + x^{-2}) dx$

$= \frac{x^3}{3} + 2x + (-x^{-1}) + C$

$= \frac{x^3}{3} + 2x - \frac{1}{x} + C$

2. $\int \frac{5x^3+1}{x^2} dx$ $\boxed{2}$

$= \int (5x + x^{-2}) dx$

$= \frac{5x^2}{2} + (-x^{-1}) + C$

$= \frac{5x^2}{2} - \frac{1}{x} + C$

3. $\int_1^4 \sqrt{2x}\, dx$ $\boxed{3}$

$= \int_1^4 (2x)^{\frac{1}{2}} dx$

$= \left[\frac{2\sqrt{2}x^{\frac{3}{2}}}{3}\right]_1^4$

$= \frac{2\sqrt{2}}{3}(8-1)$

$= \frac{14\sqrt{2}}{3}$

Section C

A.P $\quad T_n = a + (n-1)d$

1. $T_9 = 3T_6 \qquad\qquad T_4 + T_7 = -4$ $\boxed{3}$

$a + 8d = 3[a + 5d] \qquad (a+3d) + (a+6d) = -4$

$= 3a + 15d$

$2a + 7d = 0 \quad (1) \qquad 2a + 9d = -4 \quad (2)$

(2) - (1) $\quad 2d = -4$

$d = -2$

Sub into (1) $\quad 2a = 14$

$a = 7$

2. $\log_m 3 + \log_m 6 + \log_m 12 + \dots$ $\boxed{2}$

A.P. $\quad a = \log_m 3 \qquad d = \log 6 - \log_m 3$

$= \log_m 2$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2\log_m 3 + 9\log_m 2]$

$= 5[2 \times 0.48 + 9 \times 0.30]$

$= 18.3$

Section C

3. $\sum_{r=1}^{11}\left[(-1)^r 3^r + (3r+1)\right]$ $\boxed{4}$

$r=1$ $\quad -3 + 4$

$r=2$ $\quad 3^2 + 7$

$r=3$ $\quad -3^3 + 10$

$\vdots$

G.P $\qquad$ A.P

$a=-3,\ r=3$ $\qquad a=4,\ d=3$

$$S_n = \frac{a(r^n-1)}{r-1} + \frac{n}{2}\left[2a+(n-1)d\right]$$

$$S_{11} = \frac{-3((-3)^{11}-1)}{-3-1} + \frac{11}{2}\left[2(4)+10\times 3\right]$$

$$= -132\ 861 + 209$$

$$= -132\ 652$$

C 3 (alt:)

r=1	−3	+ 4	=	1
2	9	+ 7	=	+16
3	−27	+ 10	=	−17
4	+81	+ 13	=	+94
5	−243	+ 16	=	−227
6	+729	+ 19	=	+748
7	−2187	+ 22	=	−2165
8	6561	+25	=	+6586
9	−19683	+28	=	−19655
10	+59049	+31	=	+59080
11	−177147	+34	=	−177113

Section D

(i) $x^2 = 4ay$ $\boxed{3}$

$y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$= \frac{x}{2a}$

At $P(2ap, ap^2)$ $m = \frac{2ap}{2a}$

$= p$

$m_1 = -\frac{1}{p}$

Eqn of normal at P:

$y - y_1 = m(x - x_1)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

(ii) Eqn of PQ: $y = \frac{p+q}{2}x - apq$ $\boxed{2}$

Sub $S(0,a)$ $a = \frac{p+q}{2} \times 0 - apq$

$a = -apq$

$\therefore pq = -1$

(iii) Eq'n of normal at P: $\boxed{2}$

$x + py = 2ap + ap^3$ — ①

Eq'n of normal at Q:

$x + qy = 2aq + aq^3$ — ②

① − ②

$py - qy = 2ap + ap^3 - 2aq - aq^3$

$y(p-q) = 2a(p-q) + a(p^3 - q^3)$

$= 2a(p-q) + a(p-q)(p^2 + pq + q^2)$

$y = 2a + a(p^2 + pq + q^2)$
$= a(2 + p^2 + pq + q^2)$

Sub into ①

$x = 2ap + ap^3 - py$

$= 2ap + ap^3 - p[a(2 + p^2 + pq + q^2)]$

$= 2ap + ap^3 - 2ap - ap^3 - ap^2q - apq^2$

$= -apq(p+q)$

$\therefore\ N\ (-apq(p+q),\ a(2 + p^2 + pq + q^2))$

(iv) $x = -apq(p+q)$ $\boxed{4}$

$y = a(2+p^2+pq+q^2)$

Since $pq = -1$,

$x = a(p+q)$

$(p+q) = \frac{x}{a}$ ①

$y = a(2+p^2-1+q^2)$

$= a(p^2+q^2+1)$

$= a((p+q)^2-2pq+1)$

$= a((p+q)^2+2+1)$

$= a((p+q)^2+3)$

Sub in ①

$= a((\frac{x}{a})^2+3)$

$= \frac{x^2}{a}+3a$

$x^2 = a(y-3a)$

Locus of N is a vertical parabola with vertex $(0, 3a)$, focal length $\frac{1}{4}a$ and is concave up.

Section E

1. $P(x) = (x^2 - 5x + 6)Q(x) + 5(x+2)$ [2]

$P(2) = (2^2 - 5(2) + 6)Q(x) + (5(2)+2)$

$= 0 + 12 = 12$

2. $a = 2 \quad b = -1 \quad c = 1 \quad d = 1$ [3]

i) $\sum \alpha\beta = \dfrac{c}{a}$

$= \dfrac{1}{2}$

ii) $\sum \alpha\beta\gamma = \dfrac{-d}{a}$

$= -\dfrac{1}{2}$

iii) $\alpha^2\beta^2\gamma + \beta^2\gamma^2\alpha + \alpha^2\gamma^2\beta$

$= \alpha\beta\gamma[\alpha\beta + \beta\gamma + \alpha\gamma)$

$= \sum\alpha\beta\gamma \cdot \sum\alpha\beta$

$= \frac{1}{2} \cdot (-\frac{1}{2})$

$= -\frac{1}{4}$

Section F

1. $y = \log_{10} x$ $\boxed{2}$

x	1	1.5	2	2.5	3
y	0	$\log 1.5$	$\log 2$	$\log 2.5$	$\log 3$

Area $= \int_1^3 \log_{10} x \, dx$

$\doteqdot \frac{1/2}{2} \left[0 + \log 3 + 2(\log 1.5 + \log 2 + \log 2.5) \right]$

$= 0.5568$ (4 sig. figs

2. (i) $x^2 = 4y$ $\boxed{1}$

$y = \frac{x^2}{4}$ —— ①

$y^2 = 4x$ —— ②

Sub ① into ②

$\left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{16} = 4x$

$x^4 = 64x$

$x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0, \sqrt[3]{64}$

$= 0, 4$

When $x = 0$, $y = 0$

$x = 4$, $y = 4$

(iii) $y^2 = 4x$ $\boxed{3}$

$y = \sqrt{4x}$

$= 2\sqrt{x}$

$y = \frac{x^2}{4}$

$\text{Area} = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= \int_0^4 \left(2x^{\frac{1}{2}} - \frac{x^2}{4}\right) dx$

$= \left[\frac{4x^{3/2}}{3} - \frac{x^3}{12}\right]_0^4$

$= \frac{32}{3} - \frac{64}{12} - 0$

$= 5\frac{1}{3}$ units^2

Section F

3. When $x = r \quad y = \dfrac{r^2}{a}$ $\boxed{5}$

$x = 2r \quad y = \dfrac{4r^2}{a}$

$$\text{Volume} = \pi \int_{\frac{r^2}{a}}^{\frac{4r^2}{a}} (ay)\,dy$$

$$= \pi \left[\frac{ay^2}{2} \right]_{\frac{r^2}{a}}^{\frac{4r^2}{a}}$$

$$= \frac{a\pi}{2} \left[\frac{16r^4}{a^2} - \frac{r^4}{a^2} \right]$$

$$= \frac{15\pi r^2}{2a}$$

Since $V = \frac{5}{6} \pi a^3$

and $h = \dfrac{4r^2}{a} - \dfrac{r^2}{a}$

$$= \frac{3r^2}{a}$$

$$\frac{5}{6} \pi a^3 = \frac{15\pi r^4}{2a}$$

$$10\pi a^4 = 90\pi r^4$$

$$r^2 = \frac{a^2}{3}$$

$$h = \frac{3}{a}\left(\frac{a^2}{3}\right)$$
$$= a$$