NORTH SYDNEY BOYS HIGH SCHOOL

# MATHEMATICS (EXTENSION 1)

2016 HSC Course Assessment Task 4
Thursday August 18, 2016

**General instructions**

- Working time – 50 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

**SECTION I**

- Mark your answers on the answer grid provided (on page 2)

**SECTION II**

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**STUDENT NUMBER:** ........................... **# BOOKLETS USED:** .....

**Class** (please ✔)

◯ 12M3A – Mr Berry
◯ 12M3B – Ms Ziaziaris
◯ 12M3C – Miss Lee
◯ 12M4A – Mr Lin
◯ 12M4B – Dr Jomaa
◯ 12M4C – Mr Ireland

Marker's use only.

| QUESTION | 1-5 | 6 | 7 | 8 | Total | % |
|---|---|---|---|---|---|---|
| MARKS | $\overline{5}$ | $\overline{11}$ | $\overline{11}$ | $\overline{6}$ | $\overline{33}$ | |

## Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

**STUDENT NUMBER:** ..........................................

**Class** (please ✔)

◯ 12M2A – Mr Berry

◯ 12M3B – Ms Ziaziaris

◯ 12M3C – Miss Lee

◯ 12M4A – Mr Lin

◯ 12M4B – Dr Jomaa

◯ 12M4C – Mr Ireland

**1** – Ⓐ Ⓑ Ⓒ Ⓓ

**2** – Ⓐ Ⓑ Ⓒ Ⓓ

**3** – Ⓐ Ⓑ Ⓒ Ⓓ

**4** – Ⓐ Ⓑ Ⓒ Ⓓ

**5** – Ⓐ Ⓑ Ⓒ Ⓓ

# Section I

**5 marks**
**Attempt Question 1 to 5**
Mark your answers on the answer grid provided.

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**Questions** **Marks**

**1.** The correct expression for $\binom{5}{k-1}$ is **1**

(A) $\dfrac{5!}{(k-1)!(4-k)!}$

(B) $\dfrac{5!}{(4-k)!}$

(C) $\dfrac{5!}{(k-1)!(6-k)!}$

(D) $\dfrac{5!}{(6-k)!}$

**2.** A particle moving in simple harmonic motion with displacement $x$ and velocity $v$ is given by $v^2 = 9(16 - x^2)$. What is its amplitude $(A)$ and its period $(T)$? **1**

(A) $A = 3$, $T = \dfrac{\pi}{2}$

(B) $A = 3$, $T = \dfrac{2\pi}{3}$

(C) $A = 4$, $T = \dfrac{\pi}{2}$

(D) $A = 4$, $T = \dfrac{2\pi}{3}$

**3.** What is the coefficient of $x^3$ in the expansion of $(1 - 3x)^7$? **1**

(A) $-945$

(B) $-35$

(C) $35$

(D) $945$

**4.** Which of the following is an equation for simple harmonic motion? **1**

(A) $x = a \cos nt$

(B) $x = a \sin(nt - \alpha)$

(C) $x = a \cos nt - b \sin nt$

(D) All of the above

**5.** Which of the following is the coefficient of the term in $x^n$ in the expansion of $(1 + x)^n + (1 - x)^n$? **1**

(A) $0$

(B) $\dfrac{1 + (-1)^n}{2}$

(C) $1 + (-1)^n$

(D) $2$

**Examination continues overleaf...**

# Section II

**28 marks**
**Attempt Questions 6 to 8**
Write your answers in the writing booklets supplied. Additional writing booklets are available.
Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 6** (11 Marks) Commence a NEW page. **Marks**

(a) The velocity $v\ ms^{-1}$ of a particle moving in simple harmonic motion along the $x-$ axis is given by $v^2 = 8 - 2x - x^2$, where $x$ is in metres.

i. Find the acceleration of the particle in term of $x$. **1**

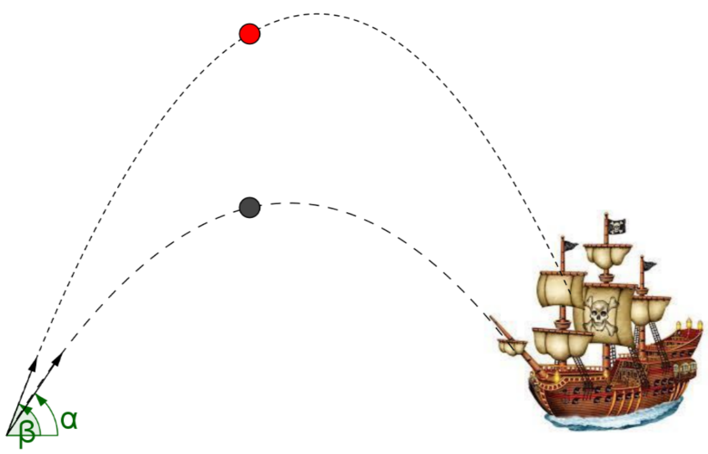
ii. State the centre and period of the motion. **2**

iii. What is the maximum speed of the particle? **1**

(b) A pirate ship is moored 560 m from a fort defending the harbor entrance of an island (see figure). The harbor defence cannon, located at sea level, has a muzzle velocity of 82 m/s. Assume that the position of the projectile is given by

$$x = vt\cos\theta, \quad y = vt\sin\theta - \frac{1}{2}gt^2,$$

where $t$ is the time, in seconds, after firing, and $g$ the acceleration due to gravity is $9.8\,ms^{-2}$.




i. Show that the angles of elevation of the cannon to hit the pirate ship are $27°$ and $63°$ correct to the nearest degrees. **3**

ii. What are the times of flight for the two elevation angles calculated above? **2**

iii. How far should the pirate ship be from the fort if it is to be beyond range of the cannon? (Hint: Maximum range corresponds to an elevation angle of $45°$.) **2**

**Please Turn Over**

**Question 7** (11 Marks) Commence a NEW page. **Marks**

(a) A particle moves in a straight line and its position at time $t$ is given by

$$x = 3 - \sqrt{3}\cos 2t + \sin 2t$$

i. Prove that the particle is moving in simple harmonic motion about $x = 3$. **3**

ii. What is the period of the motion? **1**

iii. Find the amplitude of the motion. **2**

iv. When does the particle first reach maximum speed after time $t = 0$? **2**

(b) Find the exact term independent of $x$ in the expansion of **3**

$$(1 + 2x)^2 \left(2x + \frac{1}{x}\right)^{12}$$

**Question 8** (6 Marks) Commence a NEW page. **Marks**

(a) A rocket is fired from a pontoon on the sea. The rocket is aimed at a cliff which is $h$ metres high and $d$ metres from the pontoon. The angle of projection of the rocket is $\alpha$ and its initial velocity is $v\,ms^{-1}$.

Take the point of projection as the origin O. Assume the acceleration due to gravity is $10\,ms^{-2}$. The horizontal component $x$ and vertical component $y$ of the position of the rocket at time $t$ seconds are

$$x = vt\cos\alpha, \quad y = vt\sin\alpha - 5t^2.$$

(You are NOT required to derive these.)

i. Show that the path of the rocket is given by the equation **1**

$$y = x\tan\alpha - \frac{5}{v^2}(1 + \tan^2\alpha)x^2.$$

For questions (ii) and (iii), let $d = 240$, $h = 60$, $v = 40\sqrt{2}$, and $\alpha = 45^\circ$.

ii. Find the time taken for the rocket to land on top of the cliff. **2**

iii. Find the speed of the rocket and the impact angle when it reaches this point. **3**

**End of paper.**

# Task 4 - 2016 - Ext 1

MC:

1. $\binom{5}{k-1} = \dfrac{5!}{(k-1)!\,(k-(k-1))!} = \dfrac{5!}{(k-1)!\,(6-k)!}$ (C)

2. $v^2 = 9(16-x^2)$

$\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \dfrac{d}{dx}\left(\tfrac{9}{2}(16-x^2)\right) = -9x = -n^2x \quad \therefore n=3$

$T = \dfrac{2\pi}{3}$. Also $v^2 = n^2(A^2-x^2)$, $n=3$ and $A=4$

the amplitude is 4 and the period is $\frac{2\pi}{3}$. (D)

3. $(1-3x)^7 = \sum_{k=0}^{7} \binom{7}{k} (-3x)^k = \sum_{k=0}^{7} \binom{7}{k} (-3)^k x^k$

$k=3$, the coefficient of $x^3$ is $-27\binom{7}{3} = -945$ (A)

4. (A) $x = a\cos nt$

$\dot{x} = -an\sin nt$

$\ddot{x} = -an^2\cos nt$

$= -n^2 a\cos nt$

$= -n^2x$

(B) $x = a\sin(nt-\alpha)$

$\dot{x} = an\cos(nt-\alpha)$

$\ddot{x} = -n^2 a\sin(nt-\alpha)$

$= -n^2x$

(C) $x = a\cos nt - b\sin nt$

$\dot{x} = -an\sin nt - bn\cos nt$

$\ddot{x} = -an^2\cos nt + bn^2\sin nt$

$= -n^2(a\cos nt - b\sin nt)$

$= -n^2x$

All the above (D)

5. $(1+x)^n + (1-x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{j=0}^{n} \binom{n}{j} (-x)^j$

the term in $x^n$, ($k=n$ and $j=n$) is

$\binom{n}{n} + \binom{n}{n}(-1)^n = 1 + (-1)^n$ (C)

Question 6 : (a)

(i) $v^2 = 8 - 2x - x^2 \therefore \frac{1}{2}v^2 = 4 - x - \frac{1}{2}x^2$

$\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = -1 - \frac{1}{2} \times 2x = -(1+x)$

(ii) Centre of the motion is $x = -1$ and $T = \frac{2\pi}{1} = 2\pi$

(iii) $\ddot{x} = 0 \therefore x = -1$

$\therefore v^2 = 8 - 2(-1) - (-1)^2 = 9 \therefore v = 3$.

(b) (i) $x = vt\cos\theta$, $y = vt\sin\theta - \frac{1}{2}gt^2$

$t = \frac{x}{v\cos\theta}$

$y = x\frac{\sin\theta}{\cos\theta} - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$

$y = x\frac{\sin\theta}{\cos\theta} - \frac{1}{2}g\frac{x^2}{v^2\cos^2\theta}$

$\frac{dy}{dx} = 0 \therefore \frac{\sin\theta}{\cos\theta} - g\frac{x}{v^2\cos^2\theta} = 0 \therefore \boxed{x = \frac{v^2\sin\theta\cos\theta}{g}}$

Range $= \frac{2v^2\sin\theta\cos\theta}{g} = \frac{v^2\sin 2\theta}{g}$

$560 = \frac{82^2\sin 2\theta}{9.8} \therefore \sin 2\theta = \frac{9.8 \times 560}{82^2} = 0.816$

$2\theta \simeq 54^\circ$ or $180 - 54^\circ$

$\therefore \boxed{\theta \simeq 27^\circ \text{ or } 63^\circ}$

(ii) $t_1 = \frac{x}{v\cos\theta} = \frac{560}{82\cos 27} = 7.7$ seconds

$t_2 = \frac{x}{v\cos\theta} = \frac{560}{82\cos 63} = 15$ seconds.

(iii) $\theta = 45^\circ \therefore$ Range $= \frac{v^2\sin 90^\circ}{g} = \frac{v^2}{g} = \frac{82^2}{9.8} = 690$ m

Beyond that point the ship is safe.

Question 7:

(a) $x = 3 - \sqrt{3}\cos 2t + \sin 2t$

(i) $\dot{x} = -\sqrt{3}(-2\sin 2t) + 2\cos 2t$

$= 2\sqrt{3}\sin 2t + 2\cos 2t$

$\ddot{x} = 4\sqrt{3}\cos 2t - 4\sin 2t$

$= -4(-\sqrt{3}\cos 2t + \sin 2t)$

$= -4(x-3)$

Hence the result.

(ii) $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$

(iii) $A = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

(iv) Maximum speed occurs at the centre of the motion $(x = 3)$.

$\sin 2t = \sqrt{3}\cos 2t \therefore \tan 2t = \sqrt{3}$

$2t = \pi/3$ and $t = \pi/6$.

(b) $(1+2x)^2\left(2x + \frac{1}{x}\right)^{12}$

$= (1+4x+4x^2)\sum_{k=0}^{12}\binom{12}{k}(2x)^{12-k}x^{-k}$

$= \sum_{k=0}^{12}\binom{12}{k}2^{12-k}x^{12-2k} + \sum_{k=0}^{12}4\binom{12}{k}2^{12-k}x^{13-2k} + \sum_{k=0}^{12}4\binom{12}{k}2^{12-k}x^{14-2k}$

The term independent of $x$ in $\sum_{k=0}^{12}\binom{12}{k}2^{12-k}x^{12-2k}$ corresponds to $12-2k=0$ or $k=6$, setting $k=6$ we obtain $\binom{12}{6}2^6$.

There is no term independent of $x$ in $\sum_{k=0}^{12}4\binom{12}{k}2^{12-k}x^{13-2k}$ since $k = \frac{13}{2}$ is not an integer value

The term independent of $x$ in $\sum_{k=0}^{12}4\binom{12}{k}2^{12-k}x^{14-2k}$

Corresponds to $14-2k=0$ or $k=7$, setting $k=7$ we obtain $\binom{12}{7}\times 4 \times 2^5 = \binom{12}{7}2^7$

Hence the term independent of $x$ in $(1+2x)^2\left(2x+\frac{1}{x}\right)^{12}$ is $\binom{12}{6}2^6 + \binom{12}{7}2^7$.

Question 8:

(a) (i) $t = \dfrac{x}{v\cos\alpha}$

$y = \dfrac{x}{v\cos\alpha} \cdot v\sin\alpha - 5\dfrac{x^2}{v^2\cos^2\alpha}$

$= x\tan\alpha - \dfrac{5}{v^2}x^2(1+\tan^2\alpha)$

since $\dfrac{1}{\cos^2\alpha} = \sec^2\alpha = 1+\tan^2\alpha$.

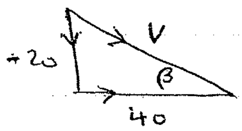
(ii) $60 = 40\sqrt{2}\sin 45\, t - 5t^2$

$5t^2 - 40t + 60 = 0$

$t^2 - 8t + 12 = 0$ or $(t-6)(t-2) = 0$

$\therefore t = 2$ or $t = 6$.

it takes 6 seconds to reach the top.

(iii) 


$\dot{x} = 40\sqrt{2} \times \dfrac{1}{\sqrt{2}} = 40$

$\dot{y} = 40\sqrt{2} \times \dfrac{1}{\sqrt{2}} - 10 \times 6 = -20$

$V^2 = (-20)^2 + 40^2 = 2000$

$V = 20\sqrt{5}$ m/s

$\tan\beta = \dfrac{20}{40} = \dfrac{1}{2} \quad \therefore \beta = 26°34'$.