NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 1)

2017 HSC Course Assessment Task 4

Friday August 18, 2017

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.
- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

- ◯ 12M3A – Ms Ziaziaris
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Hwang
- ◯ 12M3D – Mr Ireland
- ◯ 12M4A – Dr Jomaa
- ◯ 12M4B – Miss Lee
- ◯ 12M4C – Mr Lin

Marker's use only.

QUESTION	1	2	3	4	Total	%
MARKS	$\overline{10}$	$\overline{8}$	$\overline{11}$	$\overline{5}$	$\overline{34}$	

Question 1 (10 Marks) Commence a NEW page. **Marks**

When a particle moves along the x-axis with velocity v at time t, its acceleration is given by

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right).$$

The velocity $v\,ms^{-1}$ of a particle moving in simple harmonic motion along the $x-$axis is given by the expression

$$v^2 = 27 + 18x - 9x^2.$$

(a) Between which two points is the particle oscillating? **2**

(b) What is the amplitude of the motion? **1**

(c) Find the acceleration in terms of x. **2**

(d) What is the centre of the motion? **1**

(e) Find the period of the oscillation. **2**

(f) If the particle starts from the position furthest to the right, find the displacement function in terms of t. **2**

Question 2 (8 Marks) Commence a NEW page. **Marks**

A Projectile is fired from O with velocity V at an angle of inclination θ across level ground. The equations of the motion of the projectile are

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{1}{2}gt^2 \text{ (Do Not Prove this.)}$$

(a) Show that the Cartesian equation of the path can be written as **2**

$$y = \frac{-gx^2}{2V^2}\sec^2\theta + x\tan\theta.$$

(b) Show that the Cartesian equation of the path can be written as **2**

$$gx^2\tan^2\theta - 2xV^2\tan\theta + \left(2yV^2 + gx^2\right) = 0.$$

(c) Suppose that $V = 200\,ms^{-1}$, $g = 10\,ms^{-2}$ and the projectile hits a target positioned $3000\,m$ horizontally and $500\,m$ vertically from the origin. Show that $\tan\theta = \dfrac{4 \pm \sqrt{3}}{3}$. **2**

(d) Hence find the two possible values of θ correct to the nearest minute. **2**

Please Turn Over

Question 3 (11 Marks) Commence a NEW page. **Marks**

Assume that the tides rise and fall in simple harnonic motion. At low tide the channel in a harbour is $8\,m$ deep and at high tide it is $12\,m$ deep. Low tide is at $9\,a.m.$ and high tide is at $4\,p.m.$

(a) What is the depth of the channel at $12:30\,p.m.$? **1**

(b) What is the amplitude of the motion? **1**

(c) What is the period of the motion? **1**

(d) Show that the depth $y\,m$ of the water in the channel is given by $y = 10 - 2\cos\left(\frac{\pi}{7}t\right)$, where t is the number of hours after low tide. **2**

(e) Draw the graph of $y = 10 - 2\cos\left(\frac{\pi}{7}t\right)$, for $0 \leq t \leq 14$. **2**

(f) A ship needs at least $9\,m$ of water to pass through the channel safely. Mark on your graph the times when the ship can safely navigate the channel. **2**

(g) Algebraically, find the times the ship can navigate safely through the channel. **2**

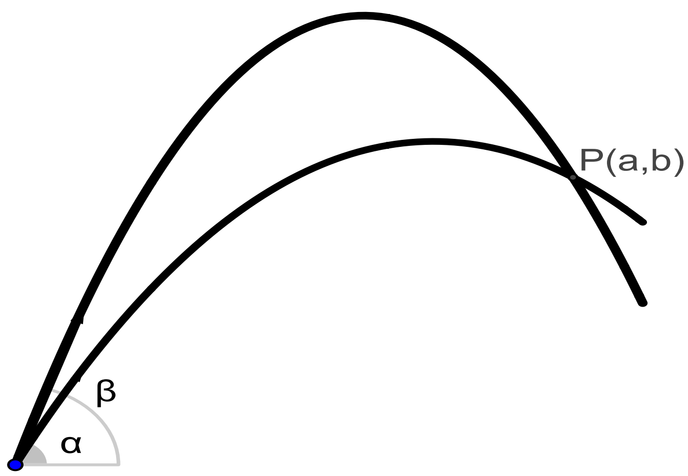
Please Turn Over

Question 4 (5 Marks) Commence a NEW page. **Marks**

The equations of the motion of the projectile are

$$x = vt\cos\theta, \quad y = vt\sin\theta - \frac{1}{2}gt^2 \text{ (Do Not Prove This)}.$$

Two particles are projected simultaneously from O in different angles of inclination with the same speed V so as to pass through another point $P(a,b)$. If α, β are the angles of projection.



(a) Prove that they pass through P at time separated by **3**

$$t_1 - t_2 = 2\frac{V}{g}\frac{\sin(\alpha-\beta)}{\cos\alpha + \cos\beta}.$$

(b) Given that **2**

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

Show that

$$t_1 - t_2 = 2\frac{V}{g}\sin\left(\frac{\alpha-\beta}{2}\right)\sec\left(\frac{\alpha+\beta}{2}\right).$$

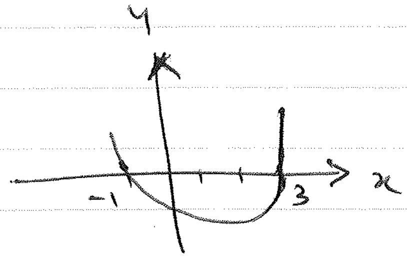
End of paper.

Question 1:

$$v^2 = 27 + 18x - 9x^2$$

(a) $v^2 = -9(x^2 - 2x - 3)$

$= -9(x-3)(x+1)$



$v^2 \geq 0 \therefore (x-3)(x+1) \leq 0$

$\therefore -1 \leq x \leq 3$

The particle is oscillating between $x = -1$ and $x = 3$.

(b) Amplitude $= \dfrac{3-(-1)}{2} = 2$

(c) $\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \dfrac{d}{dx}\left[\frac{1}{2}(27 + 18x - 9x^2)\right]$

$= \frac{1}{2}(18 - 18x)$

$= -9(x-1)$

(d) The centre of the motion is $x = 1$

(e) from (c) $n = 3$, $\therefore T = \dfrac{2\pi}{n} = \dfrac{2\pi}{3}$.

(f) $x = 1 + 2\cos(3t + \alpha)$

At $t = 0$, $x = 3$

$\therefore 3 = 1 + 2\cos(\alpha) \therefore \cos\alpha = 1 \therefore \alpha = 0$

$$\boxed{x = 1 + 2\cos 3t}$$

Question 2:

$x = Vt\cos\theta \quad , \quad y = Vt\sin\theta - \frac{1}{2}gt^2$

(a) $t = \frac{x}{V\cos\theta}$

$\therefore y = V\sin\theta \times \frac{x}{V\cos\theta} - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$

$= \frac{-gx^2}{2V^2\cos^2\theta} + \tan\theta\, x$

$= \frac{-gx^2}{2V^2}\sec^2\theta + x\tan\theta \qquad \left(\sec^2\theta = \frac{1}{\cos^2\theta}\right).$

(b) $\sec^2\theta = 1 + \tan^2\theta$

from (a) $\quad 2V^2y = -gx^2\sec^2\theta + 2V^2x\tan\theta$

$gx^2(1+\tan^2\theta) - 2V^2x\tan\theta + 2V^2y = 0$

$gx^2\tan^2\theta - 2V^2x\tan\theta + (2yV^2 + gx^2) = 0.$

(c) $V = 200\ m/s \ , \ g = 10\ m/s^2, \ x = 3\ km = 3000\ m \ , \ y = 0.5\ km = 500\ m$

From (b)

$10 \times (3000)^2\tan^2\theta - 2 \times (200)^2 \times 3000\tan\theta + 2 \times 500 \times (200)^2 + 10 \times (3000)^2 = 0$

$9\times10^7\tan^2\theta - 24\times10^7\tan\theta + 4\times10^7 + 9\times10^7 = 0$

$10^7(9\tan^2\theta - 24\tan\theta + 13) = 0$

$9\tan^2\theta - 24\tan\theta + 13 = 0$

$\Delta = 24^2 - 4\times9\times13 = 108 = 3\times6^2$

$\sqrt{\Delta} = 6\sqrt{3}$

$\tan\theta = \frac{24 \pm 6\sqrt{3}}{2\times9} = \frac{4\pm\sqrt{3}}{3}.$

(d) From (c)

$\tan\theta = \frac{4+\sqrt{3}}{3} \quad \therefore \quad \theta = 62^\circ 22'$

OR

$\tan\theta = \frac{4-\sqrt{3}}{3} \quad \therefore \quad \theta = 37^\circ 5'.$ //

Question 3:

low tide at 9:00 a.m. $y = 8$ m

High tide at 4:00 p.m. $y = 12$ m

(a) At 12:30 p.m., the depth of the channel

$$= \frac{8+12}{2} = 10 \text{ m}$$

(b) $$\text{Amplitude} = \frac{12-8}{2} = 2$$

(c) let $t=0$ at 9:00 am $\therefore$ $t=7$ at 4:00 pm

$\therefore$ period of the motion is 14 hours.

(d) $$y = 10 - 2\cos\left(\frac{\pi}{7}\right)t$$

$$\text{period} = 14 = \frac{2\pi}{n} \quad \therefore \quad n = \frac{2\pi}{14} = \frac{\pi}{7}$$

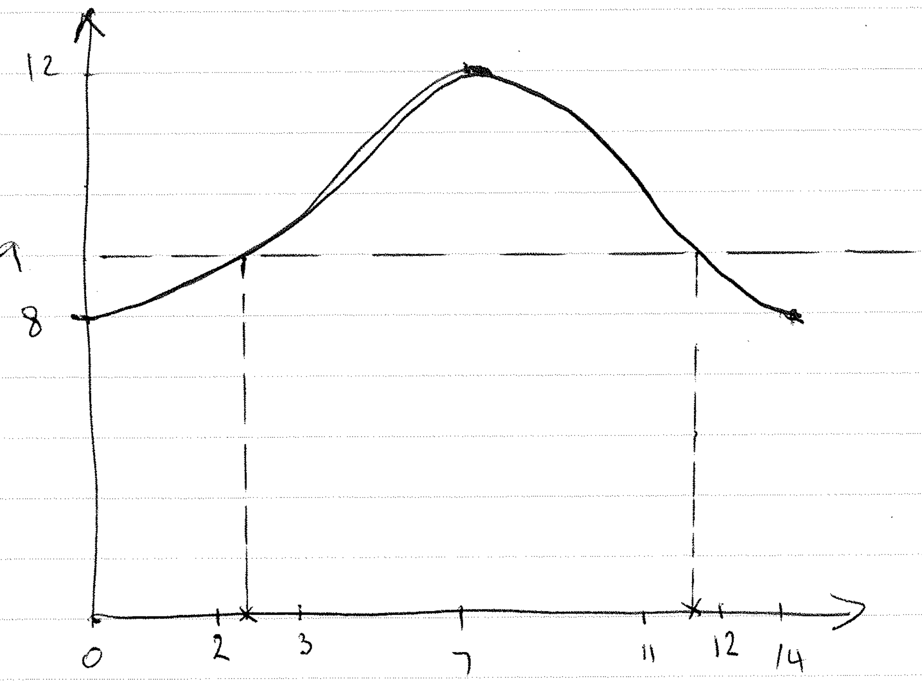
At $t=0$, $y = 10 - 2\cos 0 = 10 - 2 = 8$ (9:00 am)

At $t=7$, $y = 10 - 2\cos\pi = 10 + 2 = 12$ (4:00 pm)

At $t=3.5$, $y = 10 - 2\cos\frac{\pi}{2} = 10$ (12:30 pm)

(e)

(f)



(9) $9 = 10 - 2\cos\left(\frac{\pi}{7}t\right)$

$2\cos\left(\frac{\pi}{7}t\right) = 1 \quad \therefore \quad \cos\left(\frac{\pi}{7}t\right) = \frac{1}{2}$

$\frac{\pi}{7}t = \pi/3 \quad \text{or} \quad 5\pi/3$

$t = 7/3 \quad \text{or} \quad 35/3$

$= 2\,1/3 \quad \text{or} \quad 11\,2/3$

2 hours and 20 min or 11 hours and 40 min

Between 9: am + 2:20 = 11:20 a.m.

9: am + 11:40 = 8:40 p.m.

Question 4 : $x = vt\cos\theta$, $y = vt\sin\theta - \frac{1}{2}gt^2$

(a) $a = vt_1\cos\alpha$, $b = vt_1\sin\alpha - \frac{1}{2}gt_1^2$

$a = vt_2\cos\beta$, $b = vt_2\sin\beta - \frac{1}{2}gt_2^2$

$$vt_1\sin\alpha - \frac{1}{2}gt_1^2 = vt_2\sin\beta - \frac{1}{2}gt_2^2$$

$$vt_1\sin\alpha - vt_2\sin\beta - \frac{1}{2}g(t_1^2 - t_2^2) = 0$$

$$t_1 = \frac{a}{v\cos\alpha} , \quad t_2 = \frac{a}{v\cos\beta}$$

$$\frac{av\sin\alpha}{v\cos\alpha} - \frac{va\sin\beta}{v\cos\beta} - \frac{1}{2}g(t_1 - t_2)(t_1 + t_2) = 0$$

$$\boxed{t_1 + t_2 = \frac{a}{v}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}\right)}$$

$$a\left(\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right) - \frac{1}{2}g \times \frac{a}{v}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta}\right)(t_1 - t_2) = 0$$

$$\frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta} - \frac{g}{2v}\frac{(\cos\alpha + \cos\beta)}{\cos\alpha\cos\beta}(t_1 - t_2) = 0$$

$$t_1 - t_2 = \frac{2v}{g}\frac{\sin(\alpha - \beta)}{\cos\alpha + \cos\beta}$$

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(b) $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin(\alpha-\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$t_1 - t_2 = \frac{2v}{g} \times \frac{\cancel{2}\sin\left(\frac{\alpha-\beta}{2}\right)\cancel{\cos\left(\frac{\alpha-\beta}{2}\right)}}{\cancel{2}\cos\left(\frac{\alpha+\beta}{2}\right)\cancel{\cos\left(\frac{\alpha-\beta}{2}\right)}}$$

$$= \frac{2v}{g}\sin\left(\frac{\alpha-\beta}{2}\right)\sec\left(\frac{\alpha+\beta}{2}\right).$$

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