Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2018 HSC ASSESSMENT TASK 1

Mathematics
Extension 1

General Instructions

- Working time – 55 minutes
- Reading time – 5 minutes
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Mr Hwang
- O Mr Ireland
- O Dr Jomaa
- O Ms Lee
- O Ms Ziaziaris

Student Number: ______________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	Total	Total %
Mark	/16	/11	/10	/9	/46	/100

Question 1 (16 marks) Start a NEW page Marks

(a) Find the following integrals:

(i) $\int (2x+7)^{10}\,dx$ 2

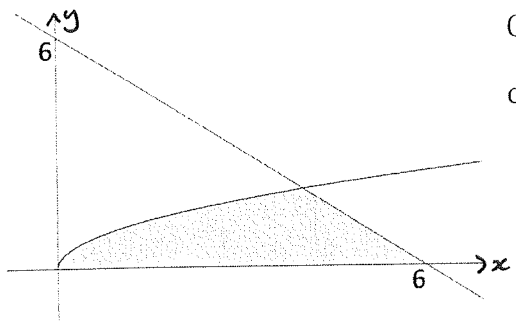
(ii) $\int \frac{5x^4-3}{x^2}dx$ 2

(iii) $\int x\sqrt{x}\,dx$ 1

(b) (i) Differentiate $(x^3+1)^5$ 1

(ii) Hence evaluate $\int_0^1 x^2(x^3+1)^4\,dx$ 2

(c)

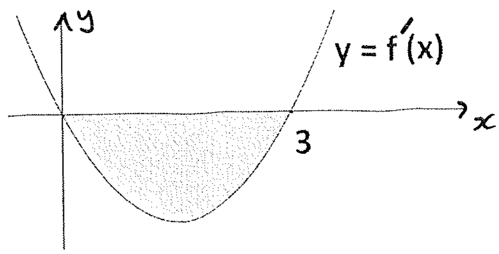


Calculate the area of the region bounded by the curves $y=\sqrt{x}$, $y=6-x$, and the x-axis. 3

(d) Use 5 function values and Simpson's Rule to approximate $\int_1^3 \frac{1}{1+x^2}dx$ 3

(answer to 2 decimal places)

(e)



The graph of $y=f'(x)$ is shown above, with the region bounded by $y=f'(x)$ and the x-axis shaded. Given that $y=f(x)$ has a local maximum at $(0,7)$ and a local minimum at $(3,4)$, find the area of the shaded region. 2

Question 2 (11 marks) Start a NEW page

(a) When a certain polynomial $P(x)$ is divided by $x-3$ the remainder is 7.

When the same polynomial is divided by $x+2$ the remainder is -3.

What is the remainder when $P(x)$ is divided by x^2-x-6 ? 3

(b) The polynomial equation $4x^3-12x^2+5x+6=0$ has 3 roots, one of which is the sum of the other two. Find all three roots. 3

(c) α, β, γ are the roots of $x^3+2x^2+3x+4=0$. Evaluate the following:

(i) $\alpha+\beta+\gamma$ 1

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma$ 1

(iii) $\alpha\beta\gamma$ 1

(iv) $\dfrac{1}{\alpha^2\beta^2}+\dfrac{1}{\beta^2\gamma^2}+\dfrac{1}{\alpha^2\gamma^2}$ 2

Question 3 (10 marks) Start a NEW page

(a) The sum of the first n terms of a series is given by the formula $\dfrac{n\,(n^2-1)}{3}$.

Find the formula for T_n, the n^{th} term of the series. 2

(b) Consider the geometric series $1+\dfrac{4}{3}\sin^2 x+\dfrac{16}{9}\sin^4 x+\dfrac{64}{27}\sin^6 x+\ldots$

(i) When the limiting sum exists, find its value in simplest form. 2

(ii) For what values of x in the interval $0<x<\dfrac{\pi}{2}$ does the limiting sum of this series exist? 2

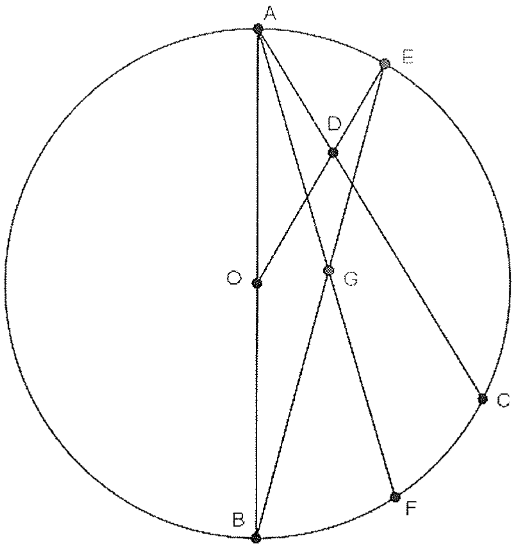
(c) The three positive numbers a, b, and c form an arithmetic progression, and $a+b+c=21$.

(i) Show that $b=7$. 1

(ii) If also the numbers $a+2,\ b+3,\ c+9$ form a geometric progression, then find c. 3

Question 4 (9 marks) Start a NEW page

(a) In the figure below, AOB is the diameter of a circle, centre O. D is a point on chord AC such that $DA = DO$, and OD is produced to E. AF is the bisector of $\angle BAC$, and cuts BE at G.



(i) Copy or trace the diagram into your answer booklet

(ii) Prove $GA = GB$ 3

(iii) Prove $AOGE$ is a cyclic quadrilateral 2

(b) Consider the polynomial $P(x) = x^3 - kx^2 + kx - 1$, where k is a real number constant.

(i) Show that 1 is a root of $P(x) = 0$. 1

(ii) Given that α, where $\alpha \neq 1$, is a second root of $P(x) = 0$, show that $\frac{1}{\alpha}$ is also a root of the equation. 1

(iii) Show that $\alpha^2 + \frac{1}{\alpha^2} = k^2 - 2k - 1$. 2

END OF EXAMINATION

Q1

(a) (i) $\int (2x+7)^{10}\,dx = \dfrac{(2x+7)^{11}}{2 \times 11} + C$

$= \dfrac{(2x+7)^{11}}{22} + C$ ✓✓ (deduct 1 for no constant)

(ii) $\int \dfrac{5x^4 - 3}{x^2}\,dx = \int (5x^2 - 3x^{-2})\,dx$

$= \dfrac{5x^3}{3} + 3x^{-1} + c$

$= \dfrac{5x^3}{3} + \dfrac{3}{x} + C$ ✓✓

(i) $\int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + C$ ✓

(b) (i) $\dfrac{d}{dx}(x^3+1)^5 = 5\cdot(x^3+1)^4\cdot 3x^2$

$= 15x^2(x^3+1)^4$ ✓

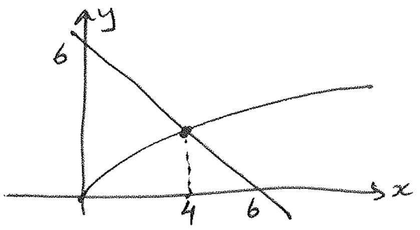
(ii) $\int_0^1 x^2(x^3+1)^4\,dx = \dfrac{1}{15}\int_0^1 15x^2(x^3+1)^4\,dx$

$= \dfrac{1}{15}\left[(x^3+1)^5\right]_0^1$ ✓

$= \dfrac{1}{15}\left[2^5 - 1\right]$

$= \dfrac{31}{15}$ ✓

Q1 (c)



At intersection,

$$\sqrt{x} = 6 - x$$
$$x = 36 - 12x + x^2$$
$$\therefore x^2 - 13x + 36 = 0$$
$$(x-9)(x-4) = 0$$
$$\therefore x = 9 \text{ or } 4$$

Here, clearly $x < 6 \therefore x = 4$. (and $y = 2$)

$$\therefore \text{Area} = \int_0^4 \sqrt{x}\,dx + \tfrac{1}{2} \times 2 \times 2$$
$$= \left[\tfrac{2}{3} x^{\frac{3}{2}}\right]_0^4 + 2$$
$$= \tfrac{2}{3} \cdot 8 + 2 \qquad \therefore \text{Area} = \tfrac{22}{3} \text{ units}^2.$$

(d)

x	1	1.5	2	2.5	3
$f(x)$	0.5	0.3077	0.2	0.1379	0.1

$$\int_1^3 \frac{1}{1+x^2}\,dx \doteq \frac{2-1}{6}\left[0.5 + 4(0.3077) + 0.2\right] + \frac{3-2}{6}\left[0.2 + 4(0.1379) + 0.1\right]$$
$$\doteq 0.46373\ldots = 0.46 \text{ (2 d.p.)}$$

[no units]

(e)

$$\text{Area} = \left|\int_0^3 f'(x)\,dx\right| = \left|\left[f(x)\right]_0^3\right|$$
$$= |4 - 7|$$
$$= 3 \text{ units}^2$$

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$\boxed{Q2}$

(a) $x^2 - x - 6 = (x-3)(x+2)$

$P(x) = (x-3)(x+2)\cdot Q(x) + ax + b$

$P(3) = 7 \quad \therefore \quad 3a + b = 7 \quad \text{----} (1)$ ✓

$P(-2) = -3 \quad \therefore \quad -2a + b = -3 \quad \text{----} (2)$ ✓

$(1)-(2) \rightarrow \quad 5a = 10 \quad \therefore \quad a = 2$

$b = 1$

$\therefore$ remainder is $2x + 1$. ✓

(b) Let roots be $\alpha, \beta, \alpha+\beta$.

Sum of roots $\rightarrow \quad \alpha + \beta + (\alpha+\beta) = \dfrac{-(-12)}{4} = 3$

$\therefore \quad \alpha + \beta = \frac{3}{2} \quad \text{------} (1)$ ✓

Product of roots $\rightarrow \quad \alpha \cdot \beta \cdot (\alpha+\beta) = -\frac{6}{4} = -\frac{3}{2}$

$\therefore \quad \alpha\beta = -1 \quad \text{------} (2)$

From (2), $\beta = -\frac{1}{\alpha}$.

Sub. into (1) $\rightarrow \quad \alpha - \frac{1}{\alpha} = \frac{3}{2}$

$\therefore \quad 2\alpha^2 - 3\alpha - 2 = 0$

$(2\alpha + 1)(\alpha - 2) = 0 \quad \therefore \quad \alpha = -\frac{1}{2}$ or 2

$\beta = 2$ or $-\frac{1}{2}$

$\alpha + \beta = 1\frac{1}{2}$

Thus roots are $2, -\frac{1}{2}, 1\frac{1}{2}$. ✓✓

$\boxed{Q2}$

(c) $x^3 + 2x^2 + 3x + 4 = 0$ has roots α, β, γ.

(i) $\alpha + \beta + \gamma = -2$ ✓

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = 3$ ✓

(iii) $\alpha\beta\gamma = -4$ ✓

(iv) $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\alpha^2\gamma^2} = \dfrac{\gamma^2 + \alpha^2 + \beta^2}{(\alpha\beta\gamma)^2}$

Now, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$$\therefore \frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\alpha^2\gamma^2} = \frac{(-2)^2 - 2(3)}{(-4)^2}$$

$$= \frac{4-6}{16}$$

$$= -\frac{1}{8}$$ ✓✓

/11

$\boxed{Q3}$

(a) $T_n = S_n - S_{n-1}$

$$= \frac{n(n^2-1)}{3} - \frac{(n-1)\left[(n-1)^2 - 1\right]}{3}$$ ✓

$$= \frac{n(n^2-1) - (n-1)(n^2 - 2n + 1 - 1)}{3}$$

$$= \frac{n^3 - n - (n-1)(n^2-2n)}{3}$$

$$= \frac{\cancel{n^3} - n - \cancel{n^3} + n^2 + 2n^2 - 2n}{3}$$

$$= \frac{3n^2 - 3n}{3} \qquad \therefore T_n = n^2 - n.$$ ✓

Q3 G.P. is $1 + \frac{4}{3}\sin^2 x + \frac{16}{9}\sin^4 x + \cdots$

(b) (i) $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{4}{3}\sin^2 x}$

$$\therefore \quad S_\infty = \frac{3}{3-4\sin^2 x}$$

(ii) S_∞ exists when $|r| < 1$

$\therefore \quad \left|\frac{4}{3}\sin^2 x\right| < 1$

$\therefore \quad \sin^2 x < \frac{3}{4} \qquad \therefore \ 0 < \sin x < \frac{\sqrt{3}}{2}$

$\therefore \ 0 < x < \frac{\pi}{3}$

(c) a, b, c is an A.P., and $a+b+c = 21$

(i) rewrite the AP as $b-d, \ b, \ b+d$

Then $(b-d) + b + (b+d) = 21$

$3b = 21$

$\therefore \ b = 7.$

(ii) The GP is $a+2, \ 10, \ c+9$ (as $b=7$)

$\therefore \quad \frac{10}{a+2} = \frac{c+9}{10} \qquad \therefore \ (a+2)(c+9) = 100$

But $c-7 = 7-a \quad \therefore \ c = 14-a.$

Subbing in, $(a+2)(23-a) = 100$

$\therefore \quad 23a - 2a + 46 - a^2 = 100$

$\therefore \quad a^2 - 21a + 54 = 0$

$(a-18)(a-3) = 0 \qquad \therefore \ a = 18$ or $a = 3.$

So the AP could be $3, 7, 11$ or $18, 7, -4.$

But all numbers $> 0 \quad \therefore$ series is $3, 7, 11$

$\therefore \ c = 11.$

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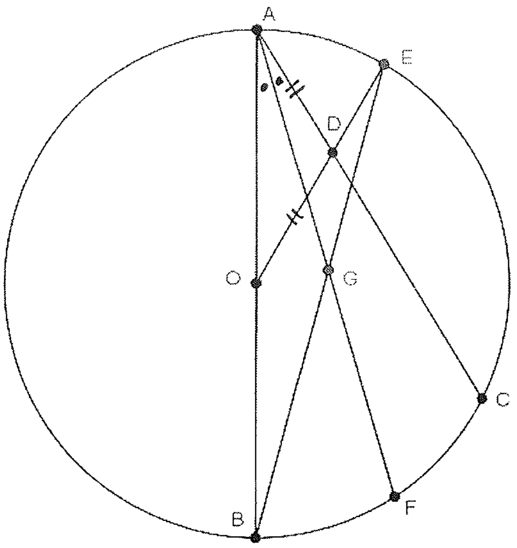
Q4

(i)



(ii) Let $\angle GBA = x$

$\therefore \angle AOE = 2x$ [angle at centre is twice the angle at the circumference standing on the same arc] ✓

But $\angle AOE = \angle OAD$ [base angles of isosceles ΔADO]

$= 2 \times \angle OAF$ [given]

$\therefore \angle OAF = x = \angle GBA$

$\therefore GA = GB$ [equal sides opposite equal angles in ΔAGB] # ✓✓

(iii) $OE = OB$ [equal radii]

$\therefore \angle OEB = \angle OBE$ [base angles of isosceles ΔOBE]

$= x$ (from part (i))

$\therefore \angle OEG = \angle OAG$

$\therefore AOGE$ is cyclic [2 points A & E on same side of interval OG subtend equal angles.] # ✓✓

Q4

(b) $P(x) = x^3 - kx^2 + kx - 1$.

(i) $P(1) = 1^3 - k\cdot 1^2 + k\cdot 1 - 1$

$= 1 - k + k - 1$

$= 0$ $\therefore$ $x=1$ is a root of $P(x)=0$. ✓

(ii) α is a root $\Rightarrow$ $\alpha^3 - k\alpha^2 + k\alpha - 1 = 0$

Now, $P\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^3 - k\left(\frac{1}{\alpha}\right)^2 + k\left(\frac{1}{\alpha}\right) - 1$

$= \frac{1}{\alpha^3} - \frac{k}{\alpha^2} + \frac{k}{\alpha} - 1$

Now $\alpha \neq 0$ (since $P(0) \neq 0$), so multiply both sides by α^3:

$\alpha^3 \cdot P\left(\frac{1}{\alpha}\right) = 1 - k\alpha + k\alpha^2 - \alpha^3$

$= -1\left(\alpha^3 - k\alpha^2 + k\alpha - 1\right)$

$= -1 \times 0$

$= 0$ since $P(\alpha) = 0$.

$\therefore$ Since $\alpha^3 \neq 0$, $P\left(\frac{1}{\alpha}\right) = 0$ $\therefore$ $\frac{1}{\alpha}$ is a root. ✓

(iii) Sum of roots is $\alpha + \frac{1}{\alpha} + 1 = k$ ✓

$\therefore$ $\alpha + \frac{1}{\alpha} = k - 1$

$\therefore$ $\left(\alpha + \frac{1}{\alpha}\right)^2 = (k-1)^2$

$\alpha^2 + \frac{1}{\alpha^2} + 2 = k^2 - 2k + 1$

$\therefore$ $\alpha^2 + \frac{1}{\alpha^2} = k^2 - 2k - 1$. ✓

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