NORTH SYDNEY BOYS HIGH SCHOOL

MATHMEMATICS EXTENSION 1

2020 HSC ASSESSMENT TASK 4
Tuesday, 18 August 2020

General instructions

- Working time – 70 min.
  (plus 5 minutes reading time)
- **Commence each new question on a new booklet**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.

Class (please ✔)

- ◯ 12MM3A2 – Ms Lee
- ◯ 12MM3B2 – Mr Hwang
- ◯ 12MM3C2 – Ms Sarofim
- ◯ 12MX2A6 – Mr Lin
- ◯ 12MX2B5 – Dr Jomaa
- ◯ 12MX2C4 – Mr Umakanthan

STUDENT NUMBER ....

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	$\overline{10}$	$\overline{16}$	$\overline{12}$	$\overline{38}$	

Question 1 (10 Marks) Commence a NEW booklet. **Marks**

(a) A projectile is launched from the top of an 50 metre building that stands on level ground. The particle's initial velocity vector is $30\vec{i}+15\vec{j}$.(Letting the ground level of the building be the origin)

Assuming acceleration due to gravity is $10ms^{-2}$

i. Express its velocity at time t in component vector form ($\vec{v}=\dot{x}\vec{i}+\dot{y}\vec{j}$). **2**

ii. Express its displacement at time t in component vector form.($\vec{r}=x\vec{i}+y\vec{j}$). **2**

iii. Find:

A. the initial speed of the projectile, **2**

B. the position vector of the projectile after 2 seconds, **2**

C. the position vector of the projectile when it reaches the ground. **2**

Question 2 (16 Marks) Commence a NEW booklet. **Marks**

(a) Use the reverse chain rule to find:

i. $\displaystyle\int \sin x \cos^5 x \, dx$ **2**

ii. $\displaystyle\int \cot 7x \, dx$ **2**

(b) Find the exact value of each definite integral.

i. $\displaystyle\int_2^3 \frac{3x}{\sqrt{x^2-2}} \, dx$ **2**

ii. $\displaystyle\int_0^{\frac{\pi}{2}} 3\sin x \; e^{\cos x} \, dx$ **3**

(c) i. Sketch the curves $y^3 = x$ and $32y = x^2$ **2**

ii. Show that the points of intersection of the curves are at $x = 0$ and $x = 8$ algebraically. **1**

iii. Show that the area formed between these curves is given by **1**

$$\int_0^8 \left(x^{\frac{1}{3}} - \frac{x^2}{32} \right) dx$$

iv. Find the volume of the solid when this region is rotated about the x-axis. **3**

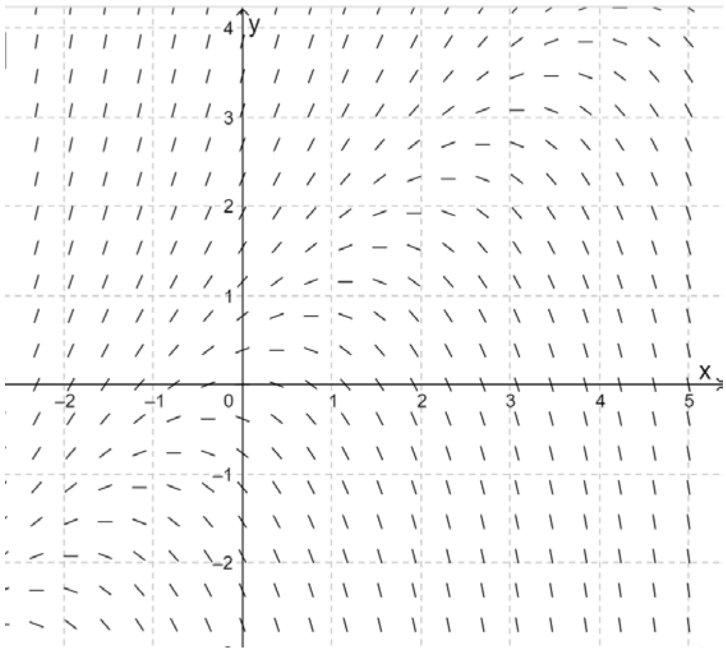
Question 3 (12 Marks) Commence a NEW booklet. **Marks**

(a) For differential equation $y' = \dfrac{5}{6x^2 + x - 1}$:

i. Show that $y' = \dfrac{3}{3x-1} - \dfrac{2}{2x+1}$. **1**

ii. Find the solution to the differential equation; given that when $x = -\dfrac{1}{4}, y = \dfrac{1}{4}$. **2**

(b) The direction field for the differential equation $\dfrac{dy}{dx} + x - y = 0$ is shown below. A solution to this differential equation that includes (2,-1) could also include **1**



A. (3,-0.2)
B. (3,-2)
C. (-1,-0.2)
D. (-1,-2)

(c) A farmer releases 20 rabbits onto a remote island off the coast of Sydney. The island is believed to support at most 5900 rabbits. The growth rate of rabbit population P is $\dfrac{dP}{dt} = kP(5900 - P)$, $P(0) = 20$ with $k > 0$ and t measured in years

i. Show that $\dfrac{1}{P(5900-P)} = \dfrac{1}{5900}\left(\dfrac{1}{P} + \dfrac{1}{5900-P}\right)$ **1**

ii. Find the equation for the population P at time t years after the release. **3**

iii. Find the value of k correct to four significant figures, if the population of the rabbits reaches 3200 after 4 years. **3**

iv. What will the population be after 9 years, correct to the nearest whole number? **1**

End of Examination

Suggested Solutions

Question 1

(a) i. (2 marks)
Initially $x = 0, y = 50,\ \dot{x} = 30, \dot{y} = 15$
$\ddot{x} = 0,\ \dot{x} = C_1$, when $t = 0$,
$\dot{x} = 30$
$\ddot{y} = -10,\ \dot{y} = -10t + C_2$, when $t = 0,\ \dot{y} = 15 = -10(0) + C2\ \ C2 = 15$
$\dot{y} = -10t + 15$

$\vec{v} = 30\vec{i} + (15 - 10t)\vec{j}$

ii. (2 marks) $\dot{x} = 30, x = 30t + C_3$
using initial condition, $t = 0, x = 0 \therefore \mathrm{C}_3 = 0$
$x = 30t$
$\dot{y} = -10y + 15, y = -5t^2 + 15t + C_4$
using initial condition, $t = 0, y = 50 \therefore \mathrm{C}_4 = 50$
$y = -5t^2 + 15t + 50$

$\vec{r} = 30t\vec{i} + (50 + 15t - 5t^2)\vec{j}$

iii. A. (2 marks)

$\vec{v} = 30\vec{i} + 15\vec{j}.$
$v^2 = 30^2 + 15^2$
$v = \sqrt{30^2 + 15^2} = 15\sqrt{5}ms^{-1}$

B. (2 marks)

$\vec{r} = 30t\vec{i} + (50 + 15t - 5t^2)\vec{j}$
when $t = 2$,
$\vec{r} = 60\vec{i} + 60\vec{j}$

C. (2 marks)

when $y = 0$
$0 = -5t^2 + 15t + 50$
$t^2 - 3t - 10 = 0$
$t = 5$ as $t > 0$
$\vec{r} = 150\vec{i}$

Question 2

(a) i. (2 marks)

$$\int \sin x \cos^5 x \, dx$$

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$dx = -\frac{du}{\sin x}$$

$$= -\int u^5 \, du$$

$$= -\frac{u^6}{6} + C$$

$$= -\frac{1}{6}\cos^6 x + C$$

ii. (2 marks)

$$\int \cot 7x \, dx$$

Let $u = 7x$

$$\frac{du}{dx} = 7$$

$$dx = \frac{du}{7}$$

$$\frac{1}{7}\int \cot u \, du$$

$$= \frac{1}{7}\int \frac{\cos u}{\sin u} \, du$$

Let $v = \sin u$

$$\frac{dv}{du} = \cos u$$

$$du = \frac{dv}{\cos u}$$

$$\frac{1}{7}\int \frac{1}{v} \, dv$$

$$= \frac{1}{7}\ln(\sin u) + C$$

$$= \frac{1}{7}\ln(\sin 7x) + C$$

(b) i. (2 marks)

$$\int_2^3 \frac{3x}{\sqrt{x^2-2}}\,dx$$

Let $u = x^2 - 2$

$\dfrac{du}{dx} = 2x$

$dx = \dfrac{du}{2x}$

upper bound $u = 2^2 - 2 = 2$

lower bound $u = 3^2 - 2 = 7$

$\dfrac{3}{2}\displaystyle\int_2^7 u^{-\frac{1}{2}}\,dx = \dfrac{3}{2} \times 2\sqrt{u}\Big|_7^2$

$=3(\sqrt{7}-\sqrt{2})u^2$

ii. (3 marks)

$$\int_0^{\frac{\pi}{2}} 3\sin x\; e^{\cos x}\,dx$$

Let $u = \cos x$

$\dfrac{du}{dx} = -\sin x$

$dx = -\dfrac{du}{\sin x}$

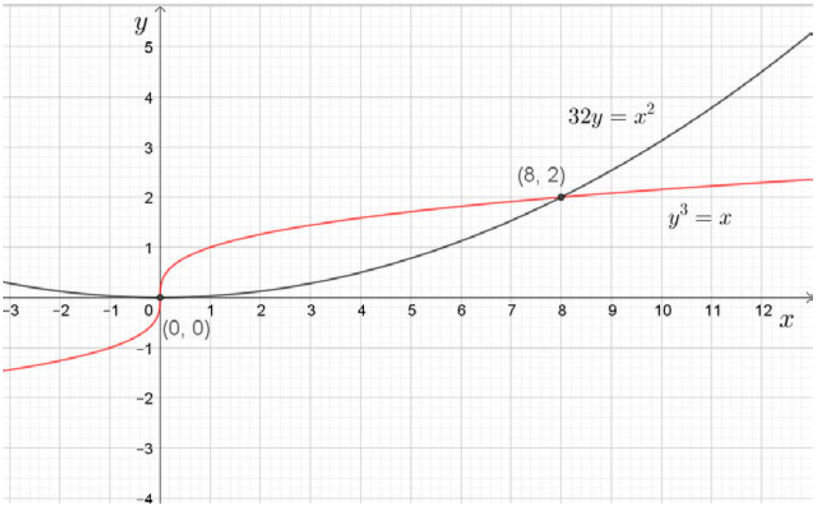
upper bound $u = 1$

lower bound $u = 0$

$-3\displaystyle\int_1^0 e^u\,dx = 3\int_0^1 e^u\,dx = 3 \times e^u\Big|_0^1$

$=3(e-1)u^2$

(c) i. (2 marks)



ii. (1 mark)

$$y = \frac{x^2}{32}$$

$$\left(\frac{x^2}{32}\right)^3 = x$$

$$\frac{x^6}{2^{15}} = x$$

$$x^6 = 2^{15}x$$

$$x(x^5 - 2^{15}) = 0$$

$$(0,0), (2^3 = 8, 2)$$

iii. (1 mark)

$$y = \frac{x^2}{32}, y = \sqrt[3]{x}$$

$y = \sqrt[3]{x}$ is above $y = \dfrac{x^2}{32}$ as shown in (i)

$$Area = \int_0^8 \left(x^{\frac{1}{3}} - \frac{x^2}{32}\right)\,dx$$

iv. (3 marks)

$$V_1 = \pi\int_0^8 x^{\frac{2}{3}}\,dx = \pi \times \left.\frac{3x^{\frac{5}{3}}}{5}\right|_0^8 = \frac{96\pi}{5}$$

$$V_2 = \pi\int_0^8 \frac{x^4}{2^{10}}\,dx = \pi \times \left.\frac{x^5}{5\times 2^{10}}\right|_0^8 = \frac{32\pi}{5}$$

$$V = \frac{96\pi}{5} - \frac{32\pi}{5} = \frac{64\pi}{5}u^3 \approx 40.21u^3 (2d.p)$$

Question 3

(a) i. (1 mark)

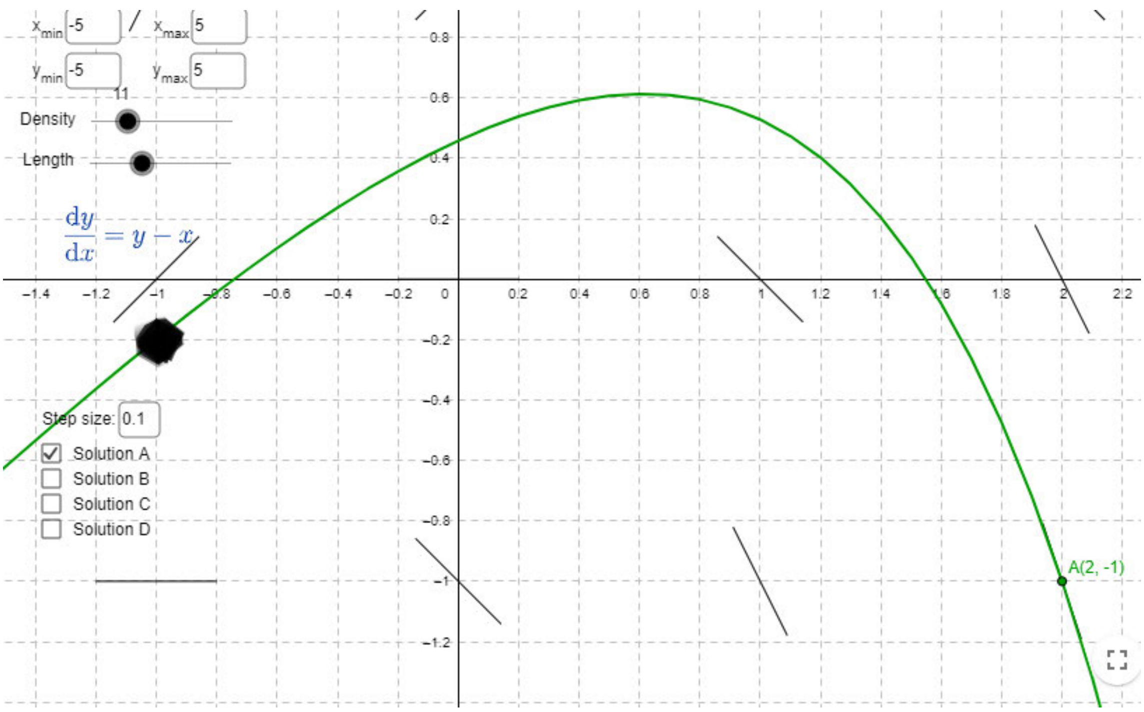
$$y' = \frac{3}{3x-1} - \frac{2}{2x+1} = \frac{3(2x+1)-2(3x-1)}{(3x-1)(2x+1)}$$
$$= \frac{5}{(3x-1)(2x+1)} \text{ as required}$$

ii. (2 marks)

$$y = \int \frac{3}{3x-1} - \frac{2}{2x+1}\,dx$$
$$y = \ln|3x-1| - \ln|2x+1| + C$$
$$y = \ln\left|\frac{3x-1}{2x+1}\right| + C$$
$$\text{sub } x = -\frac{1}{4}, y = \frac{1}{4}$$
$$C = \frac{1}{4} - \ln\left|-\frac{7}{2}\right|$$
$$\therefore \text{y=y=ln}\left|\frac{3x-1}{2x+1}\right| + \frac{1}{4} - \ln\frac{7}{2}$$

(b) (1 mark)

Draw the solution, $(-1, -0, 2)$ is on the line.
Answer is C



(c) i. (1 mark)

$$RHS = \frac{1}{5900}(\frac{1}{P} + \frac{1}{5900-P})$$
$$= \frac{1}{5900}\left(\frac{5900-P+P}{P(5900-P)}\right)$$
$$= \frac{1}{P(5900-P)} \text{ as required}$$

ii. (3 marks)

$$\frac{dP}{dt} = kP(5900-P)$$
$$\int \frac{1}{P(5900-P)}\,dP = \int k\,dt$$
$$\int \frac{1}{P} + \frac{1}{5900-P}\,dP = 5900\int k\,dt$$
$$\ln|P| - \ln|5900-P| = 5900kt + C$$
$$\ln\left|\frac{P}{5900-P}\right| = 5900kt + C$$
$$\left|\frac{P}{5900-P}\right| = Ae^{5900kt} \text{ where } A = e^C > 0$$
$$\frac{P}{5900-P} = Ae^{5900kt} \text{ where } A = \pm e^C$$
$$P = 5900Ae^{5900kt} - PAe^{5900kt}$$
$$P(1+Ae^{5900kt}) = 5900Ae^{5900kt}$$
$$P = \frac{5900Ae^{5900kt}}{1+Ae^{5900kt}} = \frac{5900}{1+Be^{-5900kt}} \text{ where } B = A^{-1}$$
$$t = 0, P = 20$$
$$20 = \frac{5900}{1+B}$$
$$B = 294$$
$$\therefore \text{P} = \frac{5900}{1+294e^{-5900kt}}$$

iii. (3 marks)

$$t = 4, P = 3200$$
$$3200 = \frac{5900}{1+294e^{(-4\times5900)k}}$$
$$k = \frac{\ln\dfrac{5900-3200}{294\times3200}}{-4\times5900}$$
$$k \approx 2.480\times10^{-4}(4s.f)$$

iv. (1 mark)

$t = 9$

$$P = \frac{5900}{1 + 294e^{(-5900\times 9)k}} = 5896.695\ldots$$

5897 rabbits