NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1
Assessment Task 3
Term 2, 2012

Name:________________________________ Mathematics Class:_______

Time Allowed: **50 minutes + 2 minutes reading time**
Total Marks: 38

Instructions:

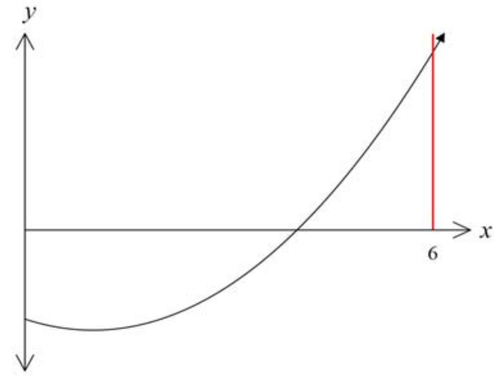
- Attempt all eight questions.
- Show all necessary working in questions 6 to 8.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.
- A table of standard integrals is provided at the back of this paper.

Question	1-1,2	1-3,4,5	2-6abc	2-6d	2-6e	2-7a	2-7b	2-7c	2-8a	2-8b	2-8c	Total
PE3		/3				/4			/2			/9
HE2					/3					/3		/6
HE6				/3				/4				/7
HE7	/2		/5				/4				/5	/16
												/38

SECTION 1. (5 marks - 1 mark each)
Multiple choice.

Write down the correct alternative on the answer sheet provided.

1. The diagram below shows the graph of $y = x^2 - 2x - 8$.



What is the correct expression for the area bounded by the x-axis and the curve $y = x^2 - 2x - 8$ between $0 \le x \le 6$?

(A) $A = \int_0^5 (x^2 - 2x - 8)dx + \left| \int_5^6 (x^2 - 2x - 8)dx \right|$

(B) $A = \int_0^4 (x^2 - 2x - 8)dx + \left| \int_4^6 (x^2 - 2x - 8)dx \right|$

(C) $A = \left| \int_0^5 (x^2 - 2x - 8)dx \right| + \int_5^6 (x^2 - 2x - 8)dx$

(D) $A = \left| \int_0^4 (x^2 - 2x - 8)dx \right| + \int_4^6 (x^2 - 2x - 8)dx$

2. $\lim\limits_{x \to 0} \dfrac{\sin 3x \cos y}{4x} =$

(A) $\dfrac{4}{3}$

(B) $\dfrac{4 \cos y}{3}$

(C) $\dfrac{3}{4}$

(D) $\dfrac{3 \cos y}{4}$

3. How many numbers greater than 3 000 can be formed from the digits 2, 3, 4, 5 and 6 if no digit is repeated?

(A) 96

(B) 120

(C) 196

(D) 216

4. How many distinct permutations of the letters of the word 'ATTAINS' are possible when the word begins and ends with the letter T?

(A) 60 (C) 120

(B) 360 (D) 1260

5. Eden, Toby and four friends arrange themselves at random in a circle. What is the probability that Eden and Toby are *not* together?

(A) $\frac{1}{120}$ (C) $\frac{2}{5}$

(B) $\frac{3}{5}$ (D) $\frac{119}{120}$

SECTION 2. (33 marks) **Marks**

- Answer each question in a separate booklet.
- Show all necessary working.

Question 6. (11 marks)

(a) Find $\int \frac{e^x}{1+e^x}\,dx$ **1**

(b) Use the table of standard integrals to evaluate $\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x dx$ **2**

(c) Find $\frac{d}{dx}\left(x\sin^2 x\right)$ **2**

(d) Find $\int \frac{x^2 dx}{(x-2)^2}$ using the substitution $u = x-2$ **3**

(e) Prove by mathematical induction that $3^{2n}-1$ is divisible by 8 for all positive integers n. **3**

Question 7. (12 marks)

(a) A team of five contestants for a quiz game is to be selected at random from a class of eight girls and five boys.

(i) What is the probability the team consists of 3 girls and 2 boys? **2**

(ii) Find the probability there are more girls chosen than boys? **2**

(Give both answers correct to 2 decimal places.)

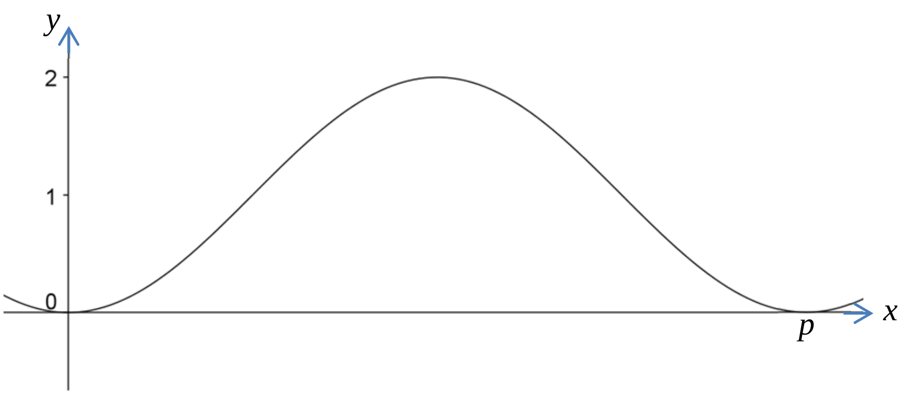
(b) Consider the function $y = x^2 e^{2x}$

(i) Show that $\frac{dy}{dx} = 2xe^{2x}(1+x)$ **1**

(ii) Find any stationary points and their nature. **3**

(iii) Sketch $y = x^2 e^{2x}$. **1**

(c)



The diagram above shows part of the graph of $y = 1 - \cos 2x$.

(i) Write down the value of p. **1**

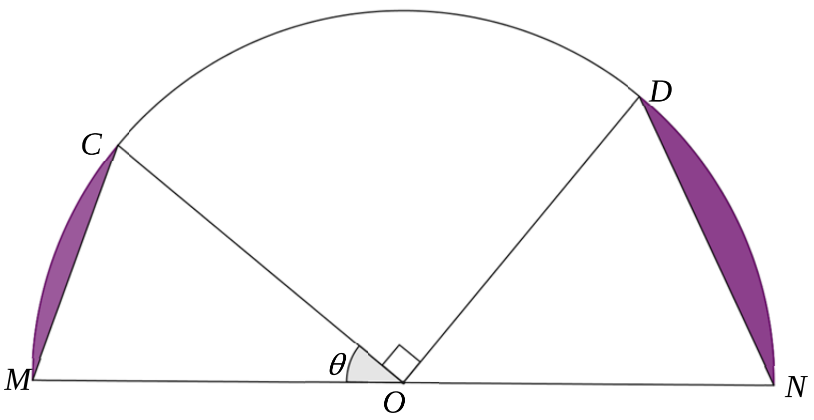
(ii) The region between $y = 1 - \cos 2x$, the x-axis, $x = 0$ and $x = p$ is rotated about the x-axis. Find the volume of the solid generated. **3**

Question 8. (10 marks)

(a) Three girls and w boys are arranged randomly in a line. Find the probability that the girls are together. Give your answer as a simplified algebraic fraction in terms of w. **2**

(b) Use the principle of mathematical induction to show that $4^n - 1 - 7n > 0$ for all integers $n \geq 2$. **3**

(c)



In the diagram MN is the diameter of a semicircle centre O and radius 1 metre.

C and D are variable points that move on the semicircle such that

$\angle MOC = \theta$ and $\angle COD = 90^0$.

(i) Show that the total area of the shaded segments A is given by

$$A = \frac{\pi}{4} - \frac{1}{2}\left(\sin\theta + \cos\theta\right)$$ **2**

(ii) Hence find the minimum shaded area A in exact form. **3**

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End of test

NSGHS Task3 X1 Maths 2012 Solutions

SECTION 1

1. x-intercept: $x^2 - 2x - 8 = 0$

$$(x-4)(x+2) = 0$$
$$x = 4, -2$$

Area between $x = 0$ and $x = 4$ negative

$$A = \left|\int_0^4 (x^2 - 2x - 8)dx\right| + \int_4^6 (x^2 - 2x - 8)dx$$

Correct answer **(D)**

2. $\lim_{x \to 0} \dfrac{\sin 3x \cos y}{4x} =$

$$\cos y \times \frac{3}{4} \times \lim_{x \to 0} \frac{\sin 3x}{3x}$$
$$= \cos y \times \frac{3}{4} \times 1$$
$$= \frac{3\cos y}{4}$$

Correct answer **(D)**

3. 5 digit numbers = 5! = 120
4 digit numbers = 4×4! = 96
Answer = 120 + 96 = 216
Correct answer **(D)**

4. T(aains)T
1×(5!÷2!)×1 ways
= 120÷2 = 60 ways
Correct answer **(A)**

5. E,T,A,B,C,D can form a ring of 6 in 5! ways - n(S)
Restriction- seat E = 1 way
seat T = 3 ways
seat rest = 4! Ways
P(not together) = (1×3×4!)÷5!
=3÷5
Correct answer **(B)**

or Regardless of where E sits, T has a choice of 5 seats, 3 0f which are not next to E. i.e. P(not together) = 3÷5

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SECTION 2

Question 6.

(a) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

$$= \ln(1+e^x) + c$$

(b)

$$\int_0^{\frac{\pi}{8}} \sec 2x \tan 2x \; dx$$
$$= \tfrac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{8}}$$
$$= \tfrac{1}{2}\left[\sec(\tfrac{\pi}{4}) - \sec(0)\right]$$
$$= \tfrac{1}{2}(\sqrt{2} - 1)$$

(c)

$$\frac{d}{dx}(x\sin^2 x)$$
$$= 1 \times \sin^2 x + x \times [2\sin x \times \tfrac{d}{dx}(\sin x)]$$
$$= \sin^2 x + 2x\sin x\cos x$$
$$= \sin x(\sin x + 2x\cos x)$$

(d)

$$\int \frac{x^2 dx + =}{(x-2)^2}$$
$$let \;\; u = x - 2$$
$$\therefore \frac{du}{dx} = 1$$
$$du = dx, x = u + 2$$
$$\int \frac{x^2 dx}{(x-2)^2} = \int \frac{(u+2)^2 du}{u^2}$$
$$= \int \frac{(u^2 + 4u + 4)du}{u^2}$$
$$= \int (1 + \frac{4}{u} + 4u^{-2})du$$
$$= u + 4\ln u - \frac{4}{u} + c$$
$$+ x - 2 + 4\ln(x-2) - \frac{4}{x-2} + c$$

(e)

When $n=1, \quad 3^{2n}-1=9-1=8$

Statement true for n = 1

Now assume statement true for $n = k$, $0 \le k \le n$

i.e. $3^{2k}-1=8J, J$ integer

$\therefore 3^{2k}=8J+1$

need to show statement true for next term i.e. $n = k + 1$

When $n=k+1, \quad 3^{2n}-1$

$$\begin{aligned} &= 3^{2(k+1)}-1 \\ &= 3^{2k+2}-1 \\ &= 3^2 \times 3^{2k}-1 \\ &= 9 \times (8J+1)-1 \\ &= 72J+8 \\ &= 8(9J+1) \end{aligned}$$

Which is divisible by 8 (common factor) true for $n = k$ +1 when true for $n = k$

Since true for n = 1 and true for next term, true for all positive integers n.

Question 7.

(a) Select 5 from 8 girls, 5 boys

${}^{13}C_5$ ways

(i) Choose 3 girls = ${}^{8}C_3$ ways

Choose 2 boys = ${}^{5}C_2$ ways

P(3 girls, 2 boys) = $\dfrac{{}^{8}C_3 \times {}^{5}C_2}{{}^{13}C_5}$

= 0.44 (2 dp)

(ii) P(More girls than boys) = (3G,2B) or (4G,1B) or (5G)

$= \dfrac{{}^{8}C_3 \times {}^{5}C_2 + {}^{8}C_4 \times {}^{5}C_1 + {}^{8}C_5}{{}^{13}C_5}$

=0.75 (2 dp)

(b)

(i)

$$y = x^2 e^{2x}$$

$$\frac{dy}{dx} = 2x \times e^{2x} + 2e^{2x} \times x^2$$

$$= 2xe^{2x}(1+x)$$

(iii) For stationary points $\dfrac{dy}{dx}=0$

$2xe^{2x}(1+x)=0$

$\therefore 2x=0, 1+x=0, e^{2x} \ne 0$

$x=0, y=0$

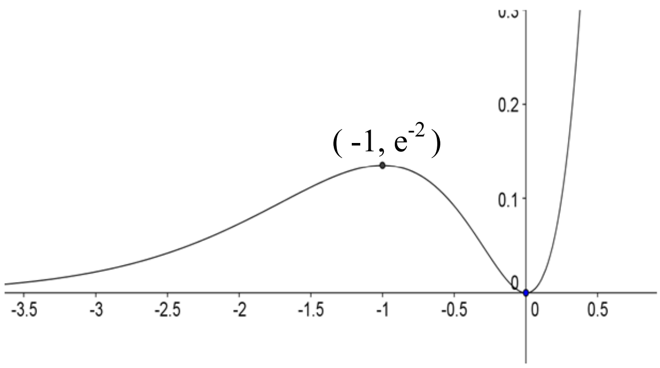
$x=-1, y=(-1)^2 e^{-2} = \dfrac{1}{e^2}$

x	-2	-1	-0.5	0	1
y'	$4e^{-2}$	0	-0.5e	0	$4e^{2}$
	+	0	-	0	+

Maximum at (-1, e^{-2})

Minimum at (0, 0)

(iii)



(c)

(i)

$1-\cos 2x = 0$

$\cos 2x = 1$

$2x = 0, 2\pi$

$x = \pi$

$p = \pi$ (first positive value)

(ii)

$$V = \pi \int_0^{\pi} (1-\cos 2x)^2 \, dx$$
$$= \pi \int_0^{\pi} (1-2\cos 2x+\cos^2 2x) \, dx$$
$$= \pi \int_0^{\pi} [1-2\cos 2x+\tfrac{1}{2}(1+\cos 4x) \,]dx$$
$$= \pi\left[x-\sin 2x+\tfrac{x}{2}+\tfrac{1}{8}\sin 4x\right]_0^{\pi}$$
$$= \pi\left[(\pi-\sin 2\pi+\tfrac{\pi}{2}+\tfrac{1}{8}\sin 4\pi)-(0-\sin 0+0-\sin 0)\right]$$
$$= \pi\left[(\pi+\tfrac{\pi}{2})-0\right]$$
$$= \frac{3\pi^2}{2}$$

Question 8.

(a) 3 girls, w boys in line

no restriction : (w + 3)! ways

girls together: 3! ways,

line of (w + 1) = (w + 1)! ways

P(girls together)

$$= \frac{(w+1)! \times 3!}{(w+3)!}$$
$$= \frac{6}{(w+3)(w+2)}$$

(b)

Prove $4^n - 1 - 7n > 0$ for $n \geq 2$

When n = 2, LHS = 4^2 – 1 - 7×2 = 1

True for n = 2

Assume true for $n = k$

i.e. 4^k -1 -7k > 0

to show true for next term

i.e. 4^{k+1}-1-7(k +1) > 0

LHS = 4^{k+1}-1-7(k +1)

= 4×4^k - 1 -7k-7

= 4×4^k -7k - 8

= 4(4^k -1 -7k) +21k - 4 > 0

Since 4^k -1 -7k > 0 by assumption

$21k - 4 > 0$ since $k \geq 2$

True for next term $n = k+1$

true for $n = k+1$ when true for $n = k$
Since true for $n = 2$ and true for next term, true for all positive integers $n \geq 2$.

(c)

(i) $\angle NOD = \dfrac{\pi}{2} - \theta$

$$\therefore A = \tfrac{1}{2}\times 1^2(\theta-\sin\theta)+\tfrac{1}{2}\times 1^2[(\tfrac{\pi}{2}-\theta)-\sin(\tfrac{\pi}{2}-\theta)]$$
$$= \tfrac{1}{2}[\theta-\sin\theta+\tfrac{\pi}{2}-\theta-\cos\theta]$$
$$= \tfrac{1}{2}[-\sin\theta+\tfrac{\pi}{2}-\cos\theta]$$
$$= \tfrac{\pi}{4}-\tfrac{1}{2}(\sin\theta+\cos\theta)$$

(ii)

$$\frac{dA}{d\theta} = 0-\tfrac{1}{2}(\cos\theta-\sin\theta)$$
$$= \tfrac{1}{2}(\sin\theta-\cos\theta)$$
$$\frac{d^2A}{d\theta^2} = \tfrac{1}{2}(\cos\theta+\sin\theta)$$

Stationary points when

$$\frac{dA}{d\theta} = \sin\theta - \cos\theta = 0$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4} \quad (\theta \text{ acute})$$
$$\frac{d^2A}{d\theta^2} = \cos\frac{\pi}{4}+\sin\frac{\pi}{4} = \frac{2}{\sqrt{2}} > 0$$

Concave up. Minimum at $x = \dfrac{\pi}{4}$

$$A_{\min} = \frac{\pi}{4}-\frac{1}{2}[\sin(\frac{\pi}{4})+\cos(\frac{\pi}{4})]$$
$$= \frac{\pi}{4}-\frac{1}{2}(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}})$$
$$= \frac{\pi}{4}-\frac{1}{\sqrt{2}}$$