NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1
Assessment Task 3
Term 2, 2013

Name:________________________________ Mathematics Class:_______

Student Number ________________________________

Time Allowed: 50 minutes + 2 minutes reading time
Total Marks: 40

Instructions:

- Attempt all eight questions.
- Show all necessary working in questions 6 to 8.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.
- A table of standard integrals is provided at the back of this paper.

Question	I q 1	I q 2, 3, 5	I Q 4	6	7a	7bc	8a	8b	Total
PE3			/1			9			/10
HE4				/12	/2				/14
HE6		/3					/3		/6
HE7	/1							/9	/10
									/40

Section I

5 marks
Attempt Questions 1 – 5
Allow about 7 minutes.
Use multiple choice answer sheet for questions 1 -5

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1. What is the value of $\lim_{x \to 0} \frac{\sin 7x}{5x}$?

(A) $\frac{7}{5}$

(B) 1

(C) $\frac{5}{7}$

(D) 0

2. What is the derivative of $y = \cos^{-1}(2x)$?

(A) $\frac{-1}{2\sqrt{1-4x^2}}$

(B) $\frac{1}{2\sqrt{1-4x^2}}$

(C) $\frac{-2}{\sqrt{1-4x^2}}$

(D) $\frac{2}{\sqrt{1-4x^2}}$

3. Which of the following is the inverse function of $f(x) = 3 - \frac{1}{x+2}$?

(A) $f^{-1}(x) = -2 - \frac{1}{3-x}$

(B) $f^{-1}(x) = 2 - \frac{1}{3-x}$

(C) $f^{-1}(x) = -2 + \frac{1}{3-x}$

(D) $f^{-1}(x) = 2 + \frac{1}{3-x}$

4. When the substitution $u = \sin x$ is used, $\int_0^{\frac{\pi}{2}} \cos x \sin^4 x \, dx$ is equivalent to

(A) $\int_0^1 u^4 \, du$

(B) $-\int_0^1 u^4 \, du$

(C) $\int_0^1 u\left(1-u^2\right)^2 du$

(D) $-\int_0^1 u\left(1-u^2\right)^2 du$

5. By using symmetry, what is the value of $\int_{-a}^{a} \cos^{-1} x \; dx$ where $-1 \le a \le 1$?

(A) 0

(B) $\dfrac{a\pi}{2}$

(C) $a\pi$

(D) $2a\pi$

End of Section I

Section II

30 marks
Attempt Questions 6 - 8
Allow about 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(12 Marks)** Use a Separate writing booklet

(a) Evaluate $\int_0^4 \frac{1}{16+x^2}$. **2**

(b) Differentiate $\sin^{-1}\left(\frac{a}{x}\right)$ with respect to x. **2**

(c) (i) What is the domain of $y=\sin^{-1}(2x-1)$? **1**

(ii) Sketch the graph of $y=\sin^{-1}(2x-1)$ **2**

(d) (i) If $\theta=\tan^{-1}A+\tan^{-1}B$ then show that $\tan\theta=\frac{A+B}{1-AB}$ **2**

(ii) Hence solve the equation $\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4}$ **3**

End of Question 6

Question 7 **(11 Marks)** Use a Separate writing booklet.

(a) Find $\lim\limits_{x \to 1} \dfrac{\left(x^2 - 1\right) + \tan\left(x - 1\right)}{x - 1}$. **2**

(b) A furnace is turned off and it begins cooling in a room at a constant temperature 22° C.
At time t minutes its temperature T decreases according to the equation

$$\frac{dT}{dt} = -k(T - 22) \text{ where } k \text{ is a positive constant.}$$

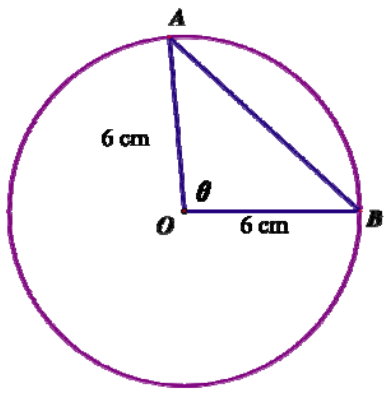
The initial temperature of the furnace is 120°C and it cools to 90° C after 20 minutes.

(i) Verify that $T = 22 + Ae^{-kt}$ is a solution to the differential equation, where A is a constant. **1**

(ii) Find the values of both A and k. **2**

(iii) How long will it take for the heater to cool to 45° C. Give your answer to the nearest minute. **2**

(c) A and B are points on the circumference of a circle with centre O and radius 6 cm.
Let $\angle AOB = \theta$.
θ is increasing at a constant rate of 0.2 radians per second.



(i) Let S be the area of the minor segment created by the chord AB in square centimetres. Show that **2**

$$\frac{dS}{d\theta} = 18\left(1 - \cos\theta\right)$$

(ii) Find the rate of change of S when $\theta = \dfrac{2\pi}{3}$ 2

End of Question 7

Question 8 **(12 Marks)** Use a Separate writing booklet

(a) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x \quad dx$ **3**

(b) Consider the function $f(x) = \frac{1}{3}\left[(x-1)^2 + 5\right]$.

(i) Sketch the parabola $y = f(x)$ showing clearly the coordinates of the vertex and any intercepts with the axes. **1**

(ii) What is the largest domain containing $x = 2$ for which the function has an inverse function $f^{-1}(x)$? **1**

(iii) State the domain of $f^{-1}(x)$ **1**

(iv) Find the coordinates of the points of intersection of the two curves $y = f^{-1}(x)$ and $y = f(x)$ **2**

(v) Sketch the graph of $y = f^{-1}(x)$ **2**

(vi) Let p be a real number such that $p < 1$ **2**
Find $f^{-1}\left(f(p)\right)$.

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Trial HSC Examination – Mathematics Extension 1 2012

Multiple Choice Answer Sheet

Name ________________________________

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Section I – Multiple Choice

$Q1$ A

$$\lim_{x\to0}\frac{\sin 7x}{5x}=\frac{7}{5}\lim_{x\to0}\frac{\sin 7x}{7x}$$
$$=\frac{7}{5}\times 1$$

$Q2$ C

$$y=\cos^{-1}(2x)$$
$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-(2x)^2}}\times 2$$
$$=\frac{-2}{\sqrt{1-4x^2}}$$

$Q3$ C

$$y=3-\frac{1}{x+2}$$

The inverse is

$$x=3-\frac{1}{y+2}$$
$$\frac{1}{y+2}=3-x$$
$$y+2=\frac{1}{3-x}$$
$$y=-2+\frac{1}{3-x}$$

$Q4$ A

$$\int_0^{\frac{\pi}{2}}\cos x\sin^4 x dx$$
$$u=\sin x$$
$$\frac{du}{dx}=\cos x$$
$$dx=\frac{du}{\cos x}$$

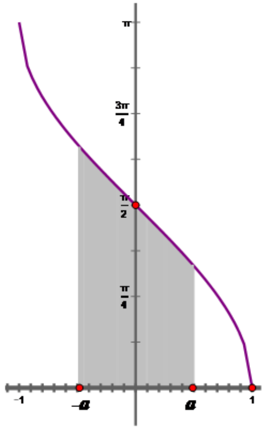
$$x=\frac{\pi}{2}$$
$$u=\sin\frac{\pi}{2}=1$$
$$x=0$$
$$u=\sin 0=0$$

$$\int_0^1 \cos x u^4\frac{du}{\cos x}$$
$$\int_0^1 u^4 du$$

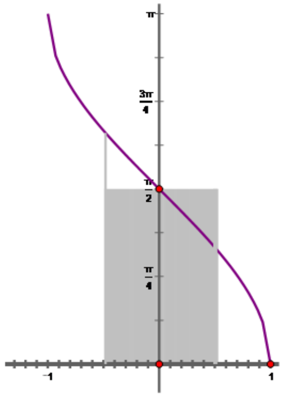
$Q5$ C

$$\int_{-a}^{a}\cos^{-1}x\ dx$$

is this area



because of the symmetry of $\cos^{-1}x$ the area is also equivalent to



The area is a rectangle

$$A=2a\times\frac{\pi}{2}=a\pi$$
$$\int_{-a}^{a}\cos^{-1}x\ dx=a\pi$$

Section II

Question 6

$a)$

$$\int_0^4 \frac{1}{16+x^2}dx = \left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_0^4$$

$$= \left(\frac{1}{4}\tan^{-1}1\right) - (0)$$

$$= \frac{1}{4}\times\frac{\pi}{4} = \frac{\pi}{16}$$

$b)$

$$\frac{d}{dx}\sin^{-1}\frac{a}{x} = \frac{1}{\sqrt{1-\frac{a^2}{x^2}}}\times\left(\frac{-a}{x^2}\right)$$

$$= \frac{-a}{x^2\sqrt{1-\frac{a^2}{x^2}}}$$

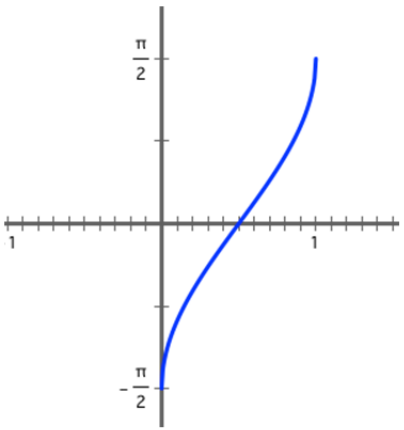
$$= \frac{-a}{x\sqrt{x^2-a^2}}$$

OR

$$= \frac{-a}{\sqrt{x^4-a^2x^2}}$$

$c)$

Domain	Range
$-1 \le 2x-1 \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$0 \le 2x \le 2$	
$0 \le x \le 1$	



$d)i)$

let $p = \tan^{-1} A$

$\therefore A = \tan p$

and $q = \tan^{-1} B$

$\therefore B = \tan q$

$\therefore \theta = p+q$

$LHS = \tan\theta$

$$= \tan(p+q)$$

$$= \frac{\tan p + \tan q}{1-\tan p \tan q}$$

$$= \frac{A+B}{1-AB}$$

$$= RHS$$

$d)ii)$

$$\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$$

$$\therefore \tan\frac{\pi}{4} = \frac{3x+2x}{1-6x^2}$$

$$1 = \frac{5x}{1-6x^2}$$

$$1-6x^2 = 5x$$

$$0 = 6x^2+5x-1$$

$$0 = (6x-1)\ (x+1)$$

$$x = \frac{1}{6} \qquad x = -1$$

But when $x = -1$

$$\tan^{-1}(-3) + \tan^{-1}(-2) \ne \frac{\pi}{4}$$

$$\therefore x = \frac{1}{6}$$

Question 7

$a)$

$$\lim_{x\to1}\frac{(x^2-1)+\tan(x-1)}{x-1}$$

$$\lim_{x\to1}\frac{(x-1)(x+1)}{x-1} + \lim_{x\to1}\frac{\tan(x-1)}{x-1}$$

$$\lim_{x\to1}(x+1) + \lim_{x\to1}\frac{\tan(x-1)}{x-1}$$

$$= (1+1)+1 = 3$$

$b)i)$

If $T = 22 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$\frac{dT}{dt} = -k(T-22)$$

$note: T - 22 = Ae^{-kt}$

$b)ii)$

When $\quad t = 0 \quad T = 120°$

$$120 = 22 + Ae^{-k(0)}$$

$$120 = 22 + A$$

$$98 = A$$

$$90 = 22 + 98e^{-20k}$$
$$\frac{68}{98} = e^{-20k}$$
$$\ln\left(\frac{68}{98}\right) = -20k$$
$$k = \frac{\ln\left(\frac{68}{98}\right)}{-20} = k = 0.0182729...$$

$b)iii)$

When $T = 45°$
$$45 = 22 + 98e^{-kt}$$
$$\frac{23}{98} = e^{-kt}$$
$$\ln\left(\frac{23}{98}\right) = -kt$$
$$\frac{\ln\left(\frac{23}{98}\right)}{-k} = t$$
$$\frac{-20\ln\left(\frac{23}{98}\right)}{\ln\left(\frac{68}{98}\right)} = t$$
$$t = 79.32327265\min$$
$$t = 79\min$$

$\frac{d\theta}{dt} = 0.2rad/s$ and $r = 6cm$
$$S = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$
$$S = \frac{1}{2}(6)^2\theta - \frac{1}{2}(6)^2\sin\theta$$
$$S = 18\theta - 18\sin\theta$$

$$\frac{dS}{d\theta} = 18\theta - 18\cos\theta$$
$$\frac{dS}{d\theta} = 18(1 - \cos\theta)$$

Find $\frac{dS}{dt}$ when $\theta = \frac{2\pi}{3}$

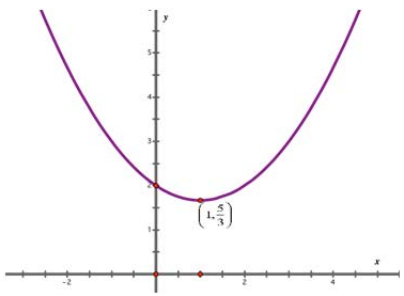
$$\frac{dS}{dt} = \frac{dS}{d\theta} \times \frac{d\theta}{dt}$$
$$\frac{dS}{dt} = 0.2 \times 18\left(1 - \cos\frac{2\pi}{3}\right)$$
$$\frac{dS}{dt} = 5.4cm^2/s$$

$Question 8$

$a)i)$
$$\int_0^{\frac{\pi}{2}} \cos^2 2x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 4x\, dx$$
$$= \frac{1}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right]$$
$$= \frac{\pi}{4}$$

$b)i)$

Vertex is $\left(1, \frac{5}{3}\right)$



$b)ii)$

Largest domain is $x \geq 1$

$biii)$

Domain is $x \geq \frac{5}{3}$

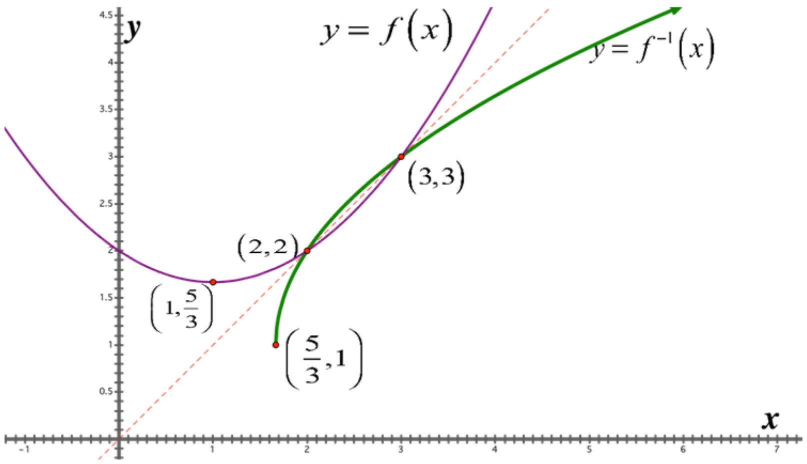
$b)iv)$

aThe inverse function $y = f^{-1}(x)$ will intersect $y = f(x)$ on the line $y = x$
$$\therefore x = \frac{1}{3}\left[(1 - x)^2 + 5\right]$$
$$3x = 1 - 2x + x^2 + 5$$
$$0 = x^2 - 5x + 6$$
$$0 = (x - 3)(x - 2)$$
$$x = 2 \qquad x = 3$$

$\therefore$ Points of intersection are

$(2,2)$ and $(3,3)$

$b)v)$



$b)vi)$

let q be the x-value such that

$f(p) = f(q)$ and $q \geq 1$

Then p and q are equidistant from $x = 1$

p is a distance of $1 - p$ to the left of $x = 1$

$\therefore$ q is a distance of $1 - p$ to the right of $x = 1$

$\therefore q = 1 + 1 - p$

$q = 2 - p$

$\therefore f(p) = f(2 - p)$

$\therefore f^{-1}(f(p)) = f^{-1}(f(2 - p)) = 2 - p$