NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1

Assessment Task 3

Term 2, 2014

Name:________________________________ **Mathematics Class:**_______

Student Number ____________________________

General Instructions:

- Reading time – 2 minutes
- Working time – 50 minutes
- Write using a black or blue pen. Black is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 6-8, show relevant mathematical reasoning and/or calculations

Section I – 5 marks
Attempt Q1-5, using the multiple choice sheet provided

Section II – 36 marks.
Attempt Questions 6-8.

Use a separate writing booklet for each question.

Submit a blank booklet if you do not attempt a question.

	Q1	Q2-5	6abc	Q6d	Q6ef	Q7a	Q7b	Q8a	Q8b	Total
HE4		/3	/6				/9	/3		/23
HE6	/1			/3		/3				/7
HE7		/1			/3				/9	/13
										/41

Section I

5 marks
Attempt Questions 1-5

Use the multiple-choice answer sheet for Questions 1-5

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1 Which expression is equal to $\int \sin^2 5x\,dx$?

(A) $\frac{1}{2}\left(x-\frac{1}{5}\sin 5x\right)+C$

(B) $\frac{1}{2}\left(x+\frac{1}{5}\sin 5x\right)+C$

(C) $\frac{1}{2}\left(x-\frac{1}{10}\sin 10x\right)+C$

(D) $\frac{1}{2}\left(x+\frac{1}{10}\sin 10x\right)+C$

2 What is the domain and range of $y=\frac{3}{2}\cos^{-1}2x$?

(A) $-2\le x\le 2,\ 0\le y\le\frac{3\pi}{2}$

(B) $-\frac{1}{2}\le x\le\frac{1}{2},\ 0\le y\le\frac{3\pi}{2}$

(C) $-2\le x\le 2,\ 0\le y\le\frac{2\pi}{3}$

(D) $-\frac{1}{2}\le x\le\frac{1}{2},\ 0\le y\le\frac{2\pi}{3}$

3 What is the derivative of $\tan^{-1}4x$?

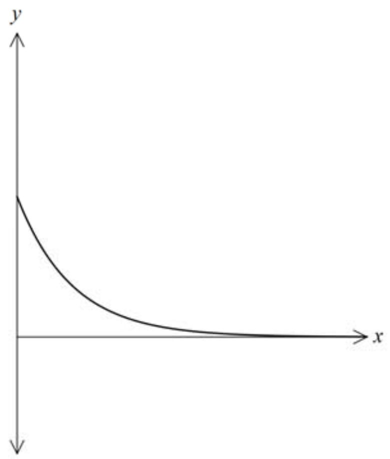
(A) $\frac{4}{1+16x^2}$ (B) $\frac{4}{16+x^2}$ (C) $\frac{2}{1+4x^2}$ (D) $\frac{2}{4+x^2}$

4 What is the exact value of $\tan^{-1}\left(\tan\dfrac{5\pi}{6}\right)$?

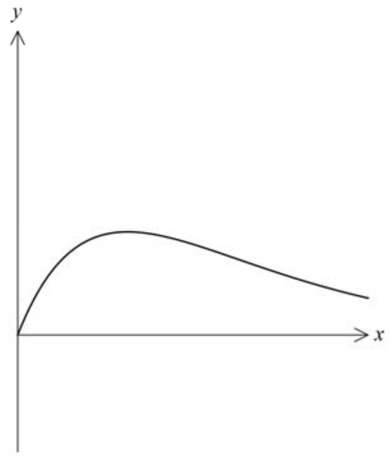
(A) $-\dfrac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\dfrac{5\pi}{6}$ (D) $-\dfrac{\pi}{6}$

5 Which of the following best represents the graph of $y = xe^{-0.5x}$ for $x \geq 0$?

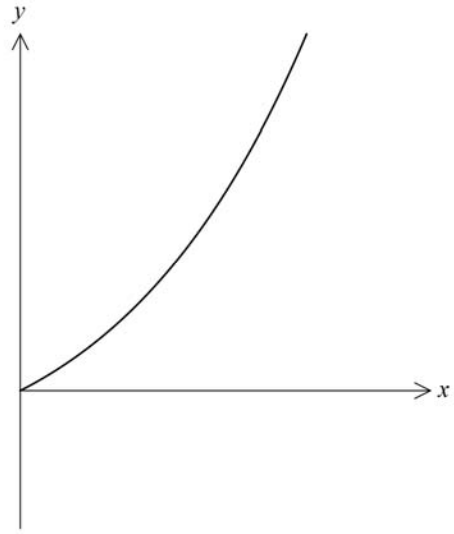
(A)



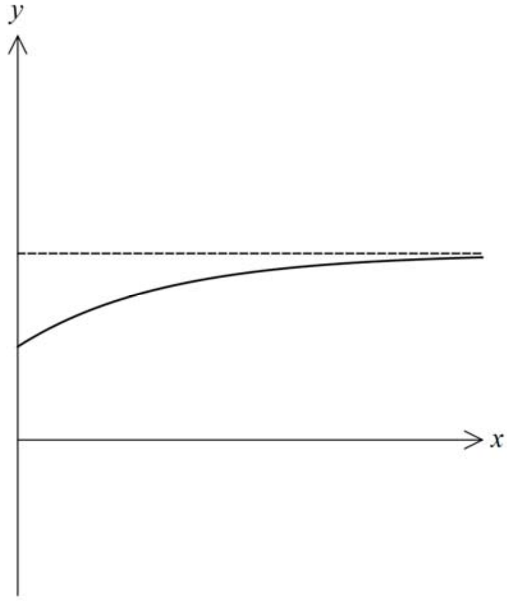
(B)



(C)



(D)



Section II

36 marks

Attempt Questions 6-8

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 marks)

(a) Find the primitive of $\dfrac{1}{9+4x^2}$ **2**

(b) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{x}{1+4x^2}\,dx$. **2**

(c) Sketch the function $y = 2\sin^{-1}(x+1)$ **2**

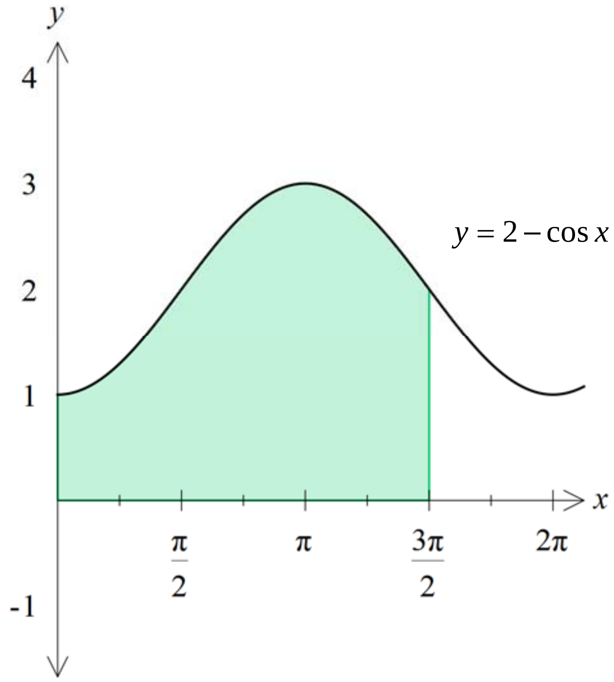
(d) Use the substitution $u = e^x$ to find $\displaystyle\int_{-\ln 2}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ **3**

(e) Write a general solution of the equation $\tan x = -\sqrt{3}$ **2**

(f) Evaluate $\displaystyle\lim_{\theta \to 0} \frac{\tan\theta}{2\theta}$ **1**

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) The region bounded by the axes and the curve $y = 2 - \cos x$ between $x = 0$ and $x = \dfrac{3\pi}{2}$ is rotated about the x axis. Find the exact volume of the solid of revolution. **3**



(b) (i) Find the equations of the horizontal and vertical asymptotes of the function **2**

$$f(x) = \frac{1}{\sqrt{4x+3}}.$$

(ii) Find the derivative of the function. **1**

(iii) On a one-third page diagram, sketch $y = f(x)$ and the line $y = x$ using the same scale on both axes. **2**

(iv) On the same diagram, sketch the inverse function $y = f^{-1}(x)$. **2**

(v) Find the equation of $y = f^{-1}(x)$, noting any restrictions on the domain. **2**

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) Find the equation of the tangent to the graph $y = \sin^{-1}(2x-1)$ at $x = \dfrac{1}{4}$. **3**

(b) (i) Show that the derivative of $f(x) = 2\left(\sqrt{x}-1\right) - \log_e x$, $x > 0$ is given by **2**

$$f'(x) = \frac{\sqrt{x}-1}{x}.$$

(ii) Find the coordinates and the nature of the stationary point on the curve. **2**

(iii) Evaluate $\lim\limits_{x\to\infty} f'(x)$. **1**

(iv) Sketch the curve, showing the features found in parts (ii) and (iii). **2**

(v) Explain why $\dfrac{\log_e x}{x} < \dfrac{2}{x}\left(\sqrt{x}-1\right)$ for $x > 1$. **2**

End of test

Multiple Choice Ext 1 - HSC Task 3 June 2014

1) $\int \sin^2 5x \, dx = \frac{1}{2}\int (1 - \cos 10x)\, dx$

$= \frac{1}{2}\left(x - \frac{1}{10}\sin 10x\right) + C$ (C)

2). $y = \frac{3}{2}\cos^{-1} 2x$

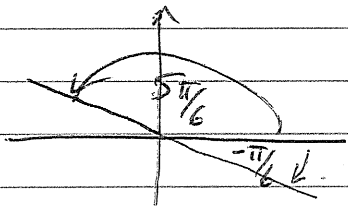
Domain : $-\frac{1}{2} \le x \le \frac{1}{2}$

Range : $0 \le y \le \frac{3\pi}{2}$ (B)

3). $\frac{d}{dx}\left[\tan^{-1}(4x)\right] = \frac{1}{1+(4x)^2} \times 4$

$= \frac{4}{1+16x^2}$ (A)

4) $\tan^{-1}\left[\tan\left(\frac{5\pi}{6}\right)\right]$



$= \tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

$= -\frac{\pi}{6}$ (D)

5) $y = xe^{-0.5x}$ for $x \ge 0$

$x = 0,\ y = 0$

$y' = x\left(-0.5e^{-0.5x}\right) + e^{-0.5x}$

$= e^{-0.5x}\left(1 - \frac{x}{2}\right)$

$\therefore y' = 0$ when $x = 2$ $\Longrightarrow$ (B)

Also, as $x \to \infty$, $e^{-0.5x} \to 0$ and it dominates $y = x$

$\therefore xe^{-0.5x} \to 0$ as $x \to \infty$.

In Summary 1) C 2) B 3) A 4) D 5) B

Question 6

a) $\int \frac{1}{9+4x^2}\,dx = \frac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx \quad a^2 = \frac{9}{4}$

$= \frac{1}{4}\times\frac{2}{3}\int \frac{3/2}{9/4+x^2}\,dx$

$= \frac{1}{6}\tan^{-1}\frac{2x}{3}$

Alternatively:

$f(x) = 2x$

$a^2 = 3$

$\int \frac{1}{a^2+[f(x)]^2} = \frac{1}{3}\times\frac{1}{2}\tan^{-1}\frac{2x}{3}$

$= \frac{1}{a}\times\frac{1}{f'(x)}\tan^{-1}\frac{f(x)}{a}$

b) $\int_0^{1/2} \frac{x}{1+4x^2}\,dx = \frac{1}{8}\int_0^{1/2}\frac{8x}{1+4x^2}$

$= \frac{1}{8}\left[\ln 1+4x^2\right]_0^{1/2}$

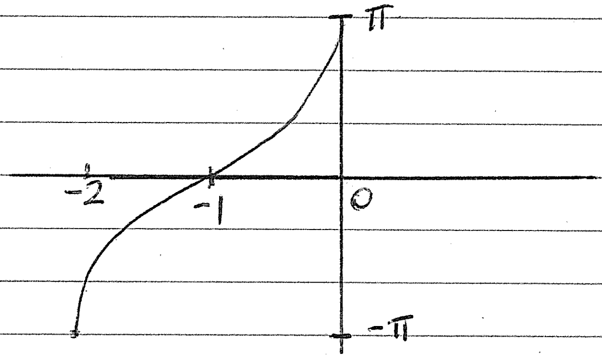
$= \frac{1}{8}(\ln 2 - \ln 1)$

$= \frac{1}{8}\ln 2$

c) $y = 2\sin^{-1}(x+1)$

$D: \quad -1 \le x+1 \le 1$

$-2 \le x \le 0$



d) $u = e^x$

$\frac{du}{dx} = e^x$

$\int_{-\ln 2}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int_{\frac{1}{2}}^{1}\frac{du}{\sqrt{1-u^2}}$

$x = 0 \Rightarrow u = 1$

$x = -\ln 2$

$= \ln \frac{1}{2} \qquad u = e^{\ln 1/2}$

$= \frac{1}{2}$

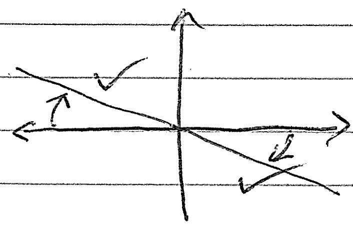
$= \left[\sin^{-1} u\right]_{1/2}^{1}$

$= \sin^{-1}(1) - \sin^{-1}(\frac{1}{2})$

$= \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$

e) $\tan x = -\sqrt{3}$



$x = n\pi - \frac{\pi}{3}$ for $n = 0, \pm1, \pm2,$ OR $n \in \mathbb{Z}$

f) $\lim_{\theta\to 0}\frac{\tan\theta}{2\theta} = \frac{1}{2}\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = \frac{1}{2}$

Question 7

$$\text{Volume} = \pi \int_0^{\frac{3\pi}{2}} (2-\cos x)^2 \, dx$$

$$= \pi \int_0^{3\pi/2} 4 - 4\cos x + \cos^2 x \, dx$$

$$= \pi \int_0^{\frac{3\pi}{2}} 4 - 4\cos x \ \frac{1}{2}(1+\cos 2x) \, dx$$

$$= \pi \left[4x - 4\sin x + \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)\right]_0^{3\pi/2}$$

$$= \pi \left[6\pi + 4 + \frac{3\pi}{4} + 0 - 0\right]$$

$$= \frac{\pi}{4}(27\pi + 16) \quad \text{or equivalent} \quad \frac{27\pi^2}{4} + 4\pi$$

$-\frac{1}{2}$ for small error at the end.

c) $f(x) = \dfrac{1}{\sqrt{4x+3}}$

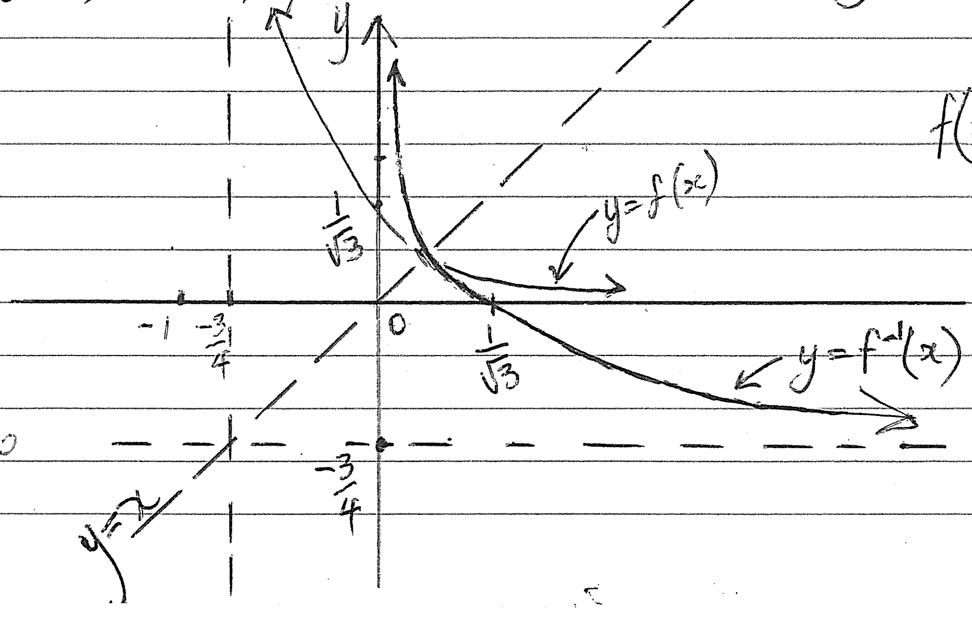
Vertical asymptote: $x = -\frac{3}{4}$

Horizontal asymptote: $y = 0$

(ii) $f'(x) = -\frac{1}{2}(4x+3)^{-3/2} \times 4$

$$= \frac{-2}{(4x+3)^{3/2}} \quad \text{or} \quad \frac{-2}{(4x+3)\sqrt{4x+3}}$$

(iii) Since $f'(x) < 0$, function is monotonically decreasing



$f(0) = \frac{1}{\sqrt{3}}$

$\doteqdot 0.5$

(iv) $y = f^{-1}(x)$ should have asymptotes $x = 0$, $y = -\frac{3}{4}$

Two graphs meet on $y = x$

(v) Let $x = \frac{1}{\sqrt{4y+3}}$

$x^2 = \frac{1}{4y+3}$

$\frac{1}{x^2} = 4y+3$

$\therefore y = \frac{1}{4}\left(\frac{1}{x^2} - 3\right)$ or $y = \frac{1-3x^2}{4x^2}$

Defined for $x > 0$ since range of $y = f(x)$ restricted to $y > 0$

<u>Question 8</u>

a) $y = \sin^{-1}(2x-1)$

$y' = \frac{1}{\sqrt{1-(2x-1)^2}} \times 2$

$= \frac{2}{\sqrt{1-(2x-1)^2}}$

At $x = \frac{1}{4}$, $y' = \frac{2}{\sqrt{3/4}} = \frac{4}{\sqrt{3}}$

Equation of tangent: $y + \frac{\pi}{6} = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$

$y = \frac{4}{\sqrt{3}}x - \frac{1}{\sqrt{3}} - \frac{\pi}{6}$

$$f(x) = 2(\sqrt{x} - 1) - \log_e x \quad , \quad x > 0$$

$$f'(x) = 2 \times \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{x}$$

$$= \frac{1}{\sqrt{x}} - \frac{1}{x}$$

$$= \frac{\sqrt{x} - 1}{x}$$

(ii) Stat points occur when $f'(x) = 0$

ie $\sqrt{x} = 1$

$x = 1$.

x	$\frac{1}{2}$	1	$1\frac{1}{2}$
$f'(x)$	-0.586	0	0.1498

There is a minimum at $(1, 0)$

Alternatively:

$$f''(x) = -\frac{1}{2} x^{-\frac{3}{2}} + \frac{1}{x^2}$$

$$= -\frac{1}{2x^{\frac{3}{2}}} + \frac{1}{x^2}$$

$$= \frac{-\sqrt{x} + 2}{2x^2}$$

$$= \frac{2 - \sqrt{x}}{2x^2}$$

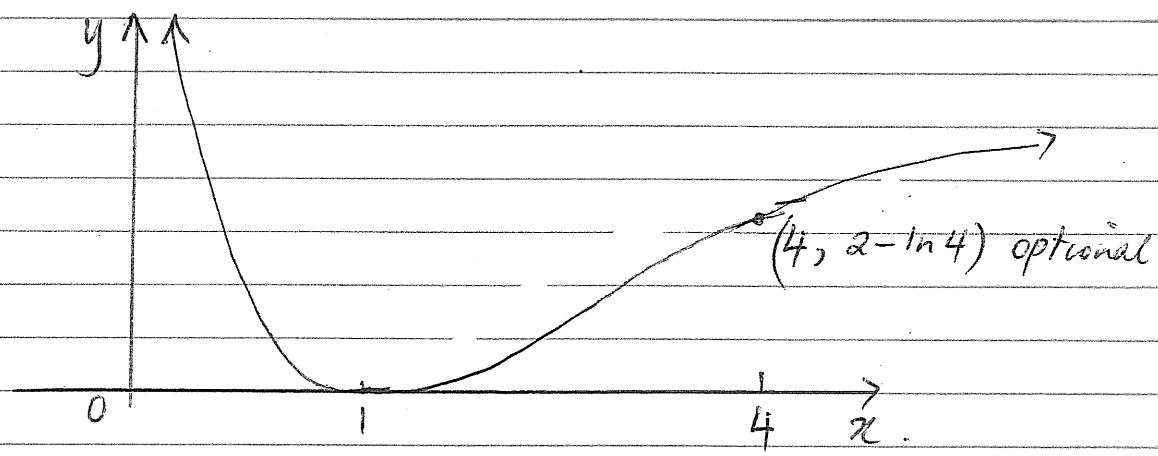
$$f''(1) = \frac{1}{2}$$

> 0

$\therefore (1, 0)$ is a minimum

(iii) $$\lim_{x \to \infty} \frac{\sqrt{x} - 1}{x} = 0$$

(iv)



(v) $2(\sqrt{x} - 1) - \log_e x > 0$ from the graph, for $x > 1$

$$2(\sqrt{x} - 1) > \log_e x$$

$$\frac{2}{x}(\sqrt{x} - 1) > \frac{\log_e x}{x}$$ since $x > 1$ inequality holds true

ie $$\frac{\log_e x}{x} < \frac{2}{x}(\sqrt{x} - 1) \text{ for } x > 1$$