NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Extension 1 Assessment Task 3
Term 2, 2015

Name:______________________________ **Mathematics Class:**_____________

Student Number: ____

General Instructions

- Reading time – 2 minutes
- Working time – 55 minutes
- Total Marks – 40 marks
- Write using a black or blue pen. Black is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 6-8, show relevant mathematical reasoning and/or calculations

Section I – 5 marks
Attempt Q1-5, using the multiple choice sheet provided

Section II – 35 marks.
Attempt Questions 6-8.

Use a separate writing booklet for each question.

Submit a blank booklet if you do not attempt a question.

	Q1–4	Q5	Q6 abc	Q6 de	Q7 a	Q7 b	Q8 a	Q8 b	Total
HE4	/4		/6			/10			**/20**
HE6				/5			/6		**/11**
HE7		/1			/2			/6	**/9**
									/40

Section I

5 marks

Attempt Questions 1 – 5

Allow about 7 minutes.

Use multiple choice answer sheet for questions 1 – 5

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1. What is the derivative of $\log_2 x$?

(A) $\dfrac{1}{x}$

(B) $\dfrac{1}{2x}$

(C) $\dfrac{\ln 2}{x}$

(D) $\dfrac{1}{x\ln 2}$

2. What is the range of the function $f(x) = \log_e\left(x^2 + e\right)$?

(A) $y \geq 0$

(B) $y > 0$

(C) $y \geq 1$

(D) $y > 1$

3. Which of the following is the derivative of $\cos^{-1}\left(e^{-x}\right)$?

(A) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$

(B) $\dfrac{-e^x}{\sqrt{1-e^{2x}}}$

(C) $\dfrac{e^{-x}}{\sqrt{1-e^{-2x}}}$

(D) $\dfrac{-e^{-x}}{\sqrt{1-e^{-2x}}}$

4. Which expression is equal to $\int \frac{1}{3+x^2}\, dx$?

(A) $\tan^{-1}\frac{x}{3}+C$

(B) $\tan^{-1}\frac{x}{\sqrt{3}}+C$

(C) $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$

(D) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$

5. What is the value of $\lim_{x\to 0}\frac{5x\cos 2x}{\sin 4x}$?

(A) 0

(B) $\frac{5}{4}$

(C) $\frac{4}{5}$

(D) Undefined

Section II

35 marks
Attempt Q6 – 8
Allow about 48 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Question 6 – 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 Marks) Use a Separate writing booklet

a) Evaluate $\int_0^1 2e^{1-x}\,dx$, in exact form. **2**

b) Find the gradient of the tangent to the graph $y = \sin^{-1}(3x-2)$ at $x = \frac{1}{2}$. **2**

c) Using the standard integral sheet, or otherwise, differentiate $y = \log_e(\sec x)$. **2**

d) Calculate the exact value of $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$. **2**

Show all necessary working.

e) Evaluate $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$, using the substitution $x = \cos\theta$, where $0 < \theta < \pi$. **3**

Leave your answer in exact form.

Question 7 (12 Marks) Use a Separate writing booklet

a) Prove that $2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}$. **2**

(Note. Using a calculator does not qualify as a proof.)

b) Let $f(x) = 1 - e^{-x^2}$ for $x \le 0$

(i) Show that the coordinates of the point of inflexion is $\left(-\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{e}}\right)$. **3**

(ii) Explain why $0 \le f(x) < 1$. **1**

(iii) Evaluate $\lim_{x \to -\infty} f'(x)$. **1**

(iv) On a one-third page diagram, sketch the graph of $y = f(x)$. **1**

(v) On the same diagram, sketch the inverse function $y = f^{-1}(x)$. **2**

(vi) Find the equation of $y = f^{-1}(x)$ noting any restrictions on the domain. **2**

Question 8 (12 Marks) Use a Separate writing booklet

a) Consider the function $f(x) = 3\cos^{-1}\frac{x}{2}$

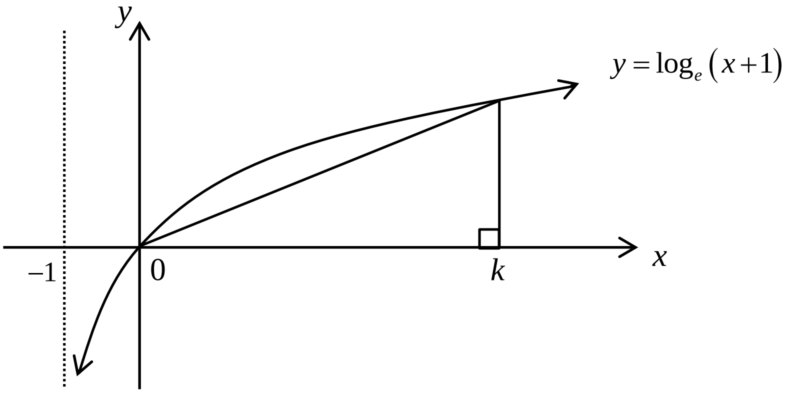
(i) Find the y intercept of the graph of the function $y = f(x)$. **1**

(ii) Sketch a graph of $y = f(x)$, showing all important features, including the intercepts and end points. **2**

(iii) Find the exact volume of the solid generated when the region bounded by the curve, the line $x = 1$ and the x-axis is rotated about the y-axis. **3**

(Question 8 continues)

b) The diagram shows the graph of $y = \log_e(x+1)$



(i) Show that the area of the region bounded by the curve $y = \log_e(x+1)$ and the x-axis for $0 \le x \le k$ is given by **3**

$$A = (k+1)\log_e(k+1) - k$$

(ii) By comparing relevant areas in the diagram show that **2**

$$\log_e(k+1) > \frac{2k}{k+2} \quad \text{for } k > 0$$

(iii) Hence, explain why $\dfrac{\log_e x}{2} > \dfrac{x-1}{x+1}$ for $x > 1$. **1**

End of test

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Answers

Section I

1 (D) 2 (C) 3 (C) 4 (D) 5 (B)

Section II

Question 6

a)

$$\int_0^1 2e^{1-x}\,dx = -2\left[e^{1-x}\right]_0^1$$
$$= -2\left(e^0 - e^1\right)$$
$$= 2(e-1)$$

b)

$$y = \sin^{-1}(3x-2)$$
$$\frac{dy}{dx} = \frac{3}{\sqrt{1-(3x-2)^2}}$$

at $x = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1-\left(\frac{3}{2}-2\right)^2}}$$
$$= \frac{3}{\sqrt{1-\frac{1}{4}}}$$
$$= 2\sqrt{3}$$

$\therefore$ the gradient of the tangent to the graph

$y = \sin^{-1}(3x-2)$ at $x = \frac{1}{2}$ is $2\sqrt{3}$

c) Standard Integral sheet:

$$\int \sec x \tan x \quad dx = \sec x$$
$$\therefore \sec x \tan x = \frac{d}{dx}\sec x$$

Now, $y = \log_e(\sec x)$

$$\therefore \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x}$$
$$\therefore \frac{dy}{dx} = \tan x$$

d)

$$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\sin\frac{\pi}{6}\right)$$
$$= \pi - \cos^{-1}\left(\sin\frac{\pi}{6}\right)$$
$$= \pi - \frac{\pi}{3}$$
$$= \frac{2\pi}{3}$$

e) Let $x = \cos\theta$

$dx = -\sin\theta\, d\theta$, where $0 < \theta < \pi$.

For $x = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ and for $x = 0$, $\theta = \frac{\pi}{2}$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sqrt{1-\cos^2\theta}} \times (-\sin\theta)\, d\theta$$
$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin\theta} \times (-\sin\theta)\, d\theta$$
$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos\theta \quad d\theta$$
$$= \left[\sin\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= \sin\frac{\pi}{2} - \sin\frac{\pi}{3}$$
$$= 1 - \frac{\sqrt{3}}{2}$$

Question 7

a) Let $A=\tan^{-1}\frac{1}{5}$ then

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

$$=\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}$$

$$=\frac{5}{12}$$

$$\therefore \tan^{-1}\frac{5}{12}=2A$$

$$\therefore \tan^{-1}\frac{5}{12}=2\tan^{-1}\frac{1}{5}$$

b)

(i)

$$f(x)=1-e^{-x^2}$$

$$f'(x)=2xe^{-x^2}$$

$$f''(x)=2e^{-x^2}+2x\times(-2x)e^{-x^2}$$

$$=2e^{-x^2}\left(1-2x^2\right)$$

$$f''(x)=0$$

$$\therefore 2e^{-x^2}\left(1-2x^2\right)=0$$

$$\therefore x=-\frac{1}{\sqrt{2}} \qquad \text{as } e^{-x^2}\neq 0$$

$$\therefore f\left(-\frac{1}{\sqrt{2}}\right)=1-e^{-\frac{1}{2}}$$

$$\therefore y=1-\frac{1}{\sqrt{e}}$$

x	-1	$-\frac{1}{\sqrt{2}}$	0
$f''(x)$	-0.36	0	2

Concavity changes

$\therefore$ the point of inflexion is

$$\left(-\frac{1}{\sqrt{2}},1-\frac{1}{\sqrt{e}}\right)$$

(ii)

As $x\to\infty, e^{-x^2}\to 0$ therefore

$1-e^{-x^2}\to 1 \qquad \therefore f(x)\to 1$

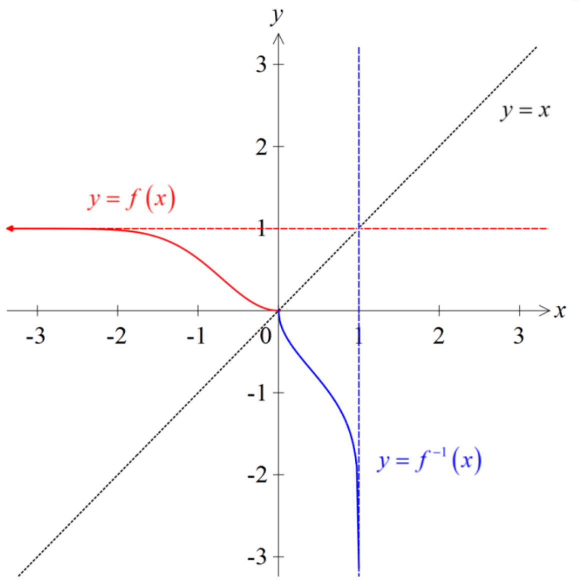
Min turning point is at $(0,0)$

$\therefore f(x)\geq 0$

Hence $0\leq f(x)<1$

(iii) $\lim\limits_{x\to-\infty} f'(x)=0$

(iv)



(v) As above

(vi)

$$x=1-e^{-y^2}$$

$$e^{-y^2}=1-x$$

$$-y^2=\log_e(1-x)$$

$$y^2=\log_e\frac{1}{(1-x)}$$

as $y\leq 0$

$$\therefore y=-\sqrt{\log_e\frac{1}{(1-x)}} \qquad \text{where } 0\leq x\leq 1$$

Question 8 (12 Marks)

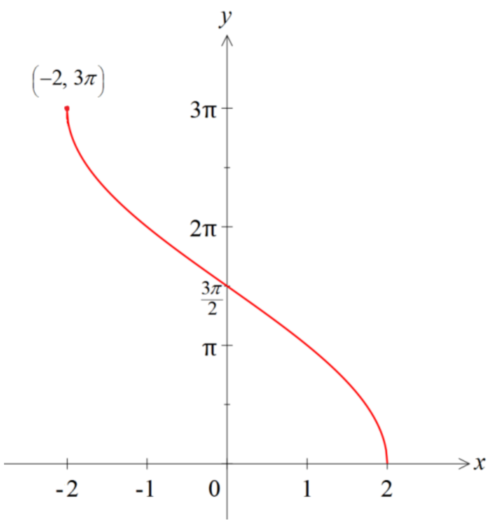
a)

(i)

$$f(x) = 3\cos^{-1}\frac{0}{2}$$
$$= 3\times\frac{\pi}{2}$$
$$= \frac{3\pi}{2}$$

$\therefore$ The y intercept of the graph of the function $y = f(x)$ is $\dfrac{3\pi}{2}$

(ii)



(iii)

$$y = 3\cos^{-1}\frac{x}{2}$$
$$\frac{y}{3} = \cos^{-1}\frac{x}{2}$$
$$\frac{x}{2} = \cos\frac{y}{3}$$
$$\therefore x = 2\cos\frac{y}{3}$$

$$V = \pi\int_0^{\pi} x^2 \quad dy - \pi\times 1^2\times\pi$$
$$= \pi\int_0^{\pi} 4\cos^2\frac{y}{3} \quad dy - \pi^2$$
$$= 2\pi\int_0^{\pi}\left(\cos\frac{2y}{3}+1\right) \quad dy - \pi^2$$
$$= 2\pi\left[\frac{3}{2}\sin\frac{2y}{3}+y\right]_0^{\pi} - \pi^2$$
$$= 2\pi\left(\frac{3}{2}\sin\frac{2\pi}{3}+\pi-0\right)-\pi^2$$
$$= 3\pi\times\frac{\sqrt{3}}{2}+2\pi^2-\pi^2$$
$$= \frac{3\pi\sqrt{3}}{2}+\pi^2$$

$$\therefore V = \pi\left(\frac{3\sqrt{3}}{2}+\pi\right) \text{ u}^3$$

b)

(i)

$$y=\log_e(x+1)$$
$$\therefore x=e^y-1$$
$$A=k\times\log_e(k+1)-\int_0^{\log_e(k+1)}\left(e^y-1\right)\ dy$$
$$=k\log_e(k+1)-\left[e^y-y\right]_0^{\log_e(k+1)}$$
$$=k\log_e(k+1)-\left(k+1-\log_e(k+1)-1\right)$$
$$=k\log_e(k+1)-\left(k-\log_e(k+1)\right)$$
$$\therefore A=(k+1)\log_e(k+1)-k$$

(ii)

$$\log_e(k+1)>\frac{2k}{k+2}\quad \text{for } k>0$$

Area of triangle is

$\frac{k}{2}\log_e(k+1)$ where $k>0$

Area of the region > Area of the triangle

$$\therefore (k+1)\log_e(k+1)-k>\frac{k}{2}\log_e(k+1)$$
$$2(k+1)\log_e(k+1)-2k>k\log_e(k+1)$$
$$(2k+2-k)\log_e(k+1)>2k$$
$$\therefore\ \log_e(k+1)>\frac{2k}{k+2}\quad \text{for } k>0$$

(iii)

For $x = k+1 \rightarrow k = x-1$

For $k>0$
$$x-1>0$$
$$x>1$$
$$\therefore \log_e(x-1+1)>\frac{2(x-1)}{x-1+2}$$
$$\log_e x>\frac{2(x-1)}{x+1}$$

Hence $\dfrac{\log_e x}{2}>\dfrac{(x-1)}{x+1}$ for $x>1$