NORTH SYDNEY GIRLS HIGH SCHOOL

2016 **HSC Assessment Task 3**

Mathematics Extension 1

General Instructions

- Reading Time – 2 minutes
- Working Time – 55 minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- In Questions 6—8, show relevant mathematical reasoning and/or calculations

Total Marks – 38

Section I Pages 2–4

5 marks

- Attempt Questions 1—5
- Allow about 8 minutes for this section

Section II Pages 5—7

33 marks

- Attempt Questions 6—8
- Allow about 47 minutes for this section

Student Name: ____________________ **Class:** ____________________

Student Number: ____________________

	MC	HE4	HE6	HE7	Total
MC	/5				/5
Q6		/5	/5	/1	/11
Q7		/11			/11
Q8			/4	/7	/11
Total	**/5**	**/16**	**/9**	**/8**	**/38**

Section I

5 marks
Attempt Questions 1–5
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1–5.

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1 Which of the following is the domain of $y = \sin^{-1} 3x$?

(A) $-3 \le x \le 3$

(B) $-\frac{1}{3} \le x \le \frac{1}{3}$

(C) $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$

(D) $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$

2 Which of the following is an expression for $\int \cos^2 5x \, dx$?

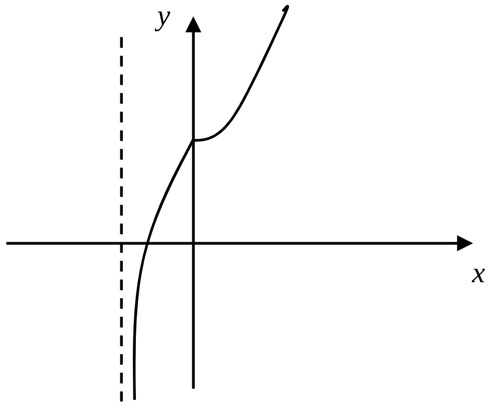
(A) $\frac{x}{2} + \frac{\sin 10x}{10} + C$

(B) $\frac{x}{2} - \frac{\sin 10x}{10} + C$

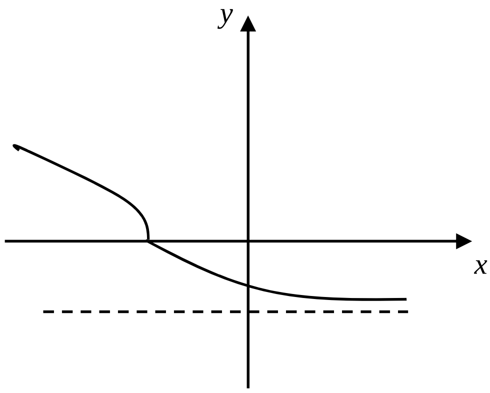
(C) $\frac{x}{2} - \frac{\sin 10x}{20} + C$

(D) $\frac{x}{2} + \frac{\sin 10x}{20} + C$

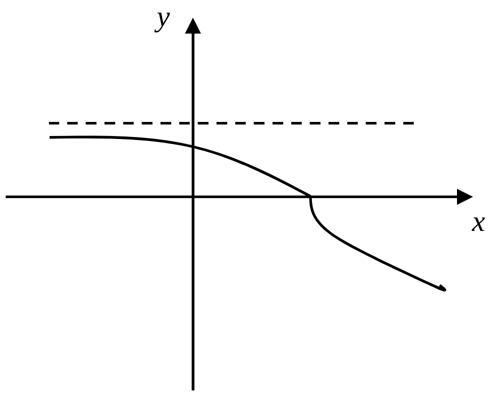
3 The graph of the function $y = f\left(x\right)$ is shown below. Which of the following could be the graph of $f^{-1}\left(x\right)$?



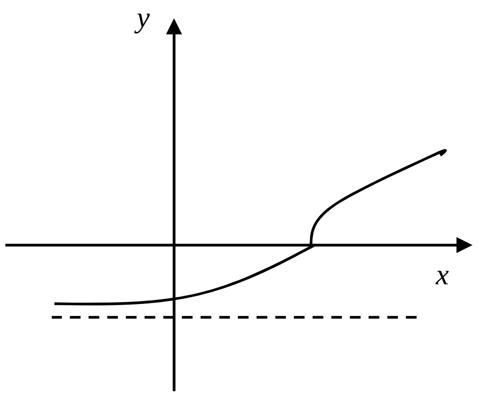
(A)



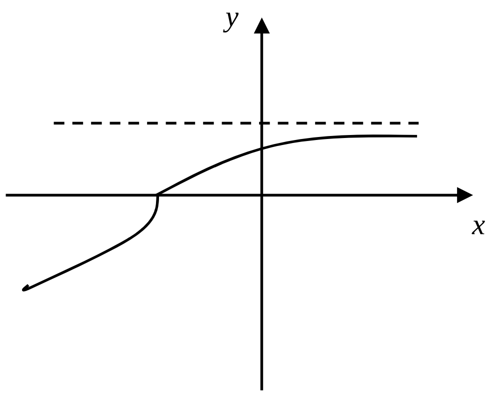
(B)



(C)

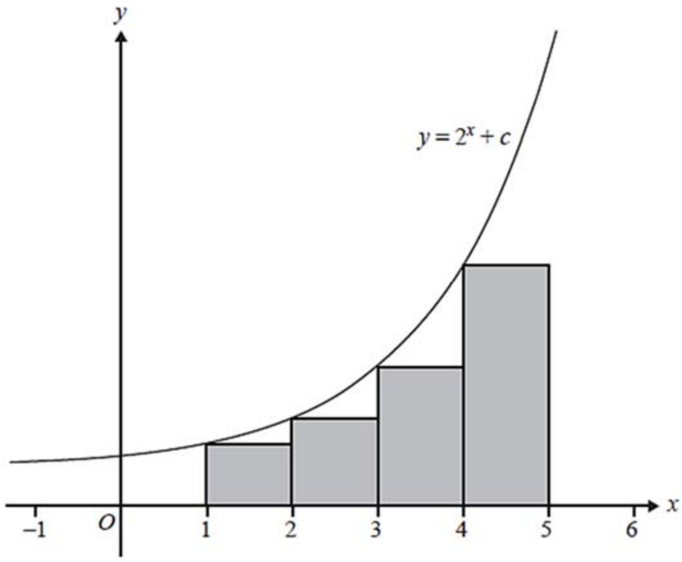


(D)



4 Consider the graph of $y = 2^x + c$, shown below, where c is a real number.

The area of the shaded rectangles is used to find an approximation to the area of the region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 5$.



If the total shaded area is 44 square units, what is the value of c?

(A) 14

(B) -4

(C) $\frac{14}{5}$

(D) $\frac{7}{2}$

5 Consider the function $f(x) = \sin\left(2x - \frac{\pi}{3}\right)$ $-a \le x \le a$. If the inverse of $f(x)$ is a function, what is the maximum possible value of a?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{6}$

(C) $\frac{\pi}{12}$

(D) $\frac{\pi}{2}$

Section II

33 marks
Attempt Questions 6–8
Allow about 47 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing papers are available.

In Questions 6–8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Use a SEPARATE writing booklet.

(a) Differentiate $\cos^{-1}\left(x^2\right)$. **2**

(b) Find the exact value of $\sin\left(\sin^{-1}\left(\frac{3}{7}\right)+\cos^{-1}\left(\frac{1}{5}\right)\right)$. **3**

(c) Evaluate $\displaystyle\int_0^{\pi} \cos^7 x \sin x \, dx$. **2**

(d) Use the substitution $u = e^{-x} + 1$ to find the indefinite integral $\displaystyle\int \frac{e^{-2x}}{e^{-x}+1}\,dx$. **3**

(e) Find $\displaystyle\lim_{x \to 0} \frac{\tan 2x}{\sin 3x}$, showing all working. **1**

Question 7 (11 marks) Use a SEPARATE writing booklet.

(a) Find a primitive function of $\dfrac{1}{9+4x^2}$. **2**

(b) (i) Write down the x-intercept of $y = 1 - \tan^{-1} x$. **1**

(ii) Sketch the graph of $y = 1 - \tan^{-1} x$ showing all important features. **2**

(c) Find the general solutions of $\cos 3x - \cos x = 0$. **2**

(d) Consider the function $f(x) = 2 + \sqrt{x}$.

(i) State the domain and range of $f(x)$. **1**

(ii) Derive the equation of $f^{-1}(x)$. **1**

(iii) Find any points of intersection of $f(x)$ and $f^{-1}(x)$. **2**

Question 8 (11 marks) Use a SEPARATE writing booklet.

(a) (i) Given $a > 1$, use the substitution $x = \dfrac{1}{u}$ to rewrite the integral **2**

$\displaystyle\int_1^a \frac{dx}{1+x^2}$ as an integral in u.

(ii) Hence, deduce the value of $\tan^{-1}(a) + \tan^{-1}\left(\dfrac{1}{a}\right)$. **2**

(b) Consider the curve $y = \log_e x$ for $x \geq 1$. A variable point P lies on the curve and the line joining P to the origin makes an angle of θ with the positive x-axis.

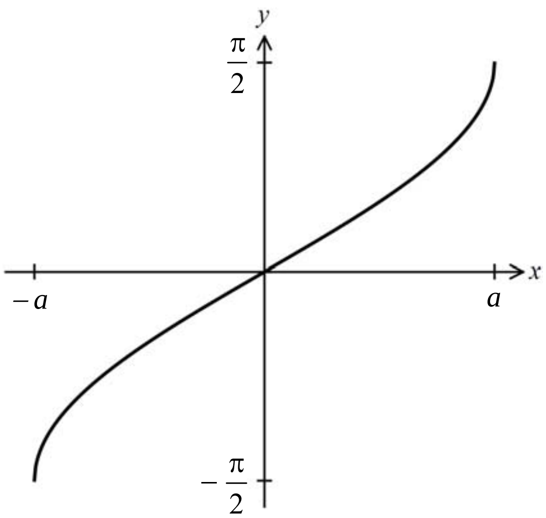
It can be shown that $\theta = \tan^{-1}\left(\dfrac{\log_e x}{x}\right)$.

(i) Show that $\dfrac{d\theta}{dx} = \dfrac{1 - \log_e x}{x^2 + \left(\log_e x\right)^2}$. **1**

(ii) Hence, find the value of x for which θ is maximum. **2**

(iii) Explain the geometrical significance of this result. **1**

(c) Consider the graph of the function $f\left(x\right) = \sin^{-1}\left(\dfrac{x}{a}\right)$ shown below. **3**



Without finding the primitive of $\sin^{-1}\left(\dfrac{x}{a}\right)$, evaluate $\displaystyle\int_{-\frac{a}{2}}^{a} \sin^{-1}\left(\dfrac{x}{a}\right) dx$

as an exact value in terms of a.

End of test

Year 12 Ext 1 HSC Assessment Task 3 : Suggested Solutions

Section I

1 (B)

$-1 \le 3x \le 1 \quad \Rightarrow \quad -\frac{1}{3} \le x \le \frac{1}{3}$

2 (D)

$$\int \cos^2 5x \, dx = \frac{1}{2}\int \left(1+\cos 10x\right) dx \qquad \text{as} \quad \cos^2 5x = \frac{1+\cos 10x}{2}$$

$$= \frac{x}{2} + \frac{\sin 10x}{20} + C$$

3 (C) reflect in the line $y = x$

4 (D)

$$\text{Area rectangles} = \left(2^1 + c\right) + \left(2^2 + c\right) + \left(2^3 + c\right) + \left(2^4 + c\right)$$

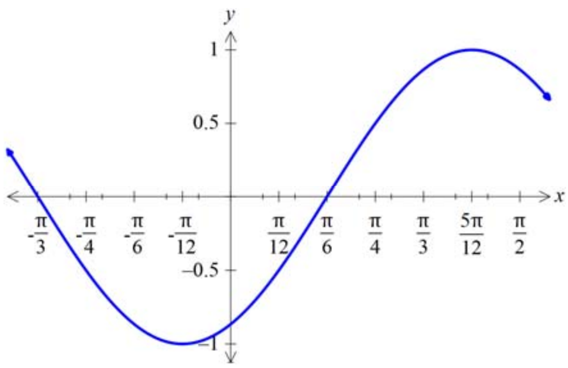
$$= \left(2+4+8+16\right) + 4c = 30 + 4c$$

$30 + 4c = 44$

$c = \frac{14}{4} = \frac{7}{2}$

5 (C)

Consider the graph of $y = \sin\left(2x - \frac{\pi}{3}\right)$.



To have an inverse which is a function, the graph cannot turn in the domain. As the graph turns at $x = -\frac{\pi}{12}$ and $\frac{5\pi}{12}$, the max value of a is $\frac{\pi}{12}$.

Section II

Question 6 (11 marks) Use a SEPARATE writing booklet.

(a)	Differentiate $\cos^{-1}\left(x^2\right)$.	**2**

$$\frac{d}{dx}\left(\cos^{-1}\left(x^2\right)\right) = \frac{-1}{\sqrt{1-\left(x^2\right)^2}} \times 2x$$

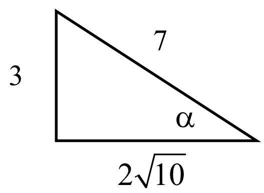
$$= \frac{-2x}{\sqrt{1-x^4}}$$

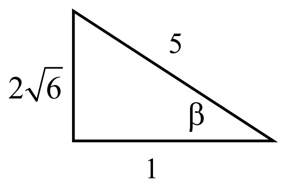
Generally done well.

Some forgot to use the chain rule, others didn't square x^2.

(b)	Find the exact value of $\sin\left(\sin^{-1}\left(\frac{3}{7}\right)+\cos^{-1}\left(\frac{1}{5}\right)\right)$.	**3**

Let $\sin^{-1}\left(\frac{3}{7}\right) = \alpha$ and $\cos^{-1}\left(\frac{1}{5}\right) = \beta$





$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{3}{7}.\frac{1}{5} + \frac{2\sqrt{6}}{5}.\frac{2\sqrt{10}}{7}$$

$$= \frac{3}{35} + \frac{4\sqrt{60}}{7}$$

$$= \frac{3+8\sqrt{15}}{35}$$

The most common error was to substitute the ratios into the wrong place in the formula.

Others didn't recognise that the expansion of $\sin(A+B)$ was required.

A small number did not recall the expansion correctly.

(c) Evaluate $\int_0^{\pi} \cos^7 x \sin x \, dx$. **2**

$$\int_0^{\pi} \cos^7 x \sin x dx = -\int_0^{\pi} \cos^7 x(-\sin x) dx$$
$$= -\left[\frac{\cos^8 x}{8}\right]_0^{\pi}$$
$$= -(1-1)$$
$$= 0$$

Generally done well, but a significant number could not substitute correctly, particularly with regard to the negative that arises.

Many did not recognise this standard question type.

(d) Use the substitution $u = e^{-x} + 1$ to find the indefinite integral $\int \frac{e^{-2x}}{e^{-x}+1} dx$. **3**

$$\int \frac{e^{-2x}}{e^{-x}+1} dx = \int \frac{e^{-x}.e^{-x}}{e^{-x}+1} dx$$
$$= \int \frac{(u-1).(-du)}{u}$$
$$= \int \frac{1-u}{u} du$$
$$= \int \left(\frac{1}{u} - 1\right) du$$
$$= \log_e u - u + C$$
$$= \log_e (e^{-x}+1) - (e^{-x}+1) + C$$

Let $u = e^{-x} + 1$

$$\frac{du}{dx} = -e^{-x}$$
$$e^{-x} dx = -du$$

Many students used over-complicated algebra in setting up their integral. You should not need to have x and u in the same integral. In fact, if this had been a **definite** integral you would have been penalised for doing that because the limits of integration cannot match **both** x and u .

Many had issues with the sign. If your sign was wrong as a result of setting up the integral incorrectly you lost a full mark. If you made a subsequent sign error after getting a correct integrand you lost half a mark.

A number left their final answer in terms of u . Others did not recognise the log integral.

(e) Find $\lim\limits_{x\to 0}\dfrac{\tan 2x}{\sin 3x}$, showing all working. **1**

$$\lim_{x\to 0}\frac{\tan 2x}{\sin 3x}=\frac{2}{3}\times\lim_{x\to 0}\frac{\tan 2x}{2x}\times\lim_{x\to 0}\frac{3x}{\sin 3x}$$
$$=\frac{2}{3}\times 1\times 1=\frac{2}{3}$$

Splitting $\dfrac{\tan 2x}{\sin 3x}$ into $\dfrac{2}{3}\cdot\dfrac{\tan 2x}{2}\dfrac{3}{\sin 3x}$ was not sufficient because the limit rules do not apply to those expressions. On the other hand, you should not write $\dfrac{2x}{3x}\cdot\lim\limits_{x\to 0}\dfrac{\tan 2x}{2x}\dfrac{3x}{\sin 3x}$, as $\dfrac{2x}{3x}$ also needs to be evaluated in the limit. The first error was penalised, the second was not.

Question 7 (11 marks) Use a SEPARATE writing booklet.

(a) Find a primitive function of $\dfrac{1}{9+4x^2}$. **2**

$$\int\frac{dx}{9+4x^2}=\frac{1}{4}\int\frac{dx}{\frac{9}{4}+x^2}$$
$$=\frac{1}{4}.\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)+C$$
$$=\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

This question was completed quite well. The two main methods were to either:

- Factor out the 4 from the denominator $\dfrac{1}{4}\times\dfrac{1}{\frac{9}{4}+x^2}$ and then using the standard integral $\displaystyle\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ from the sheet. The most common error here was to forget to multiply by $\dfrac{1}{a}$ which is $\dfrac{2}{3}$.
- Write the question in the form $\dfrac{f'(x)}{a^2+\left[f(x)\right]^2}$. The most common error in this method was to not consider $f'(x)$ $\dfrac{1}{2}\times\dfrac{2}{3^2+[2x]^2}$.

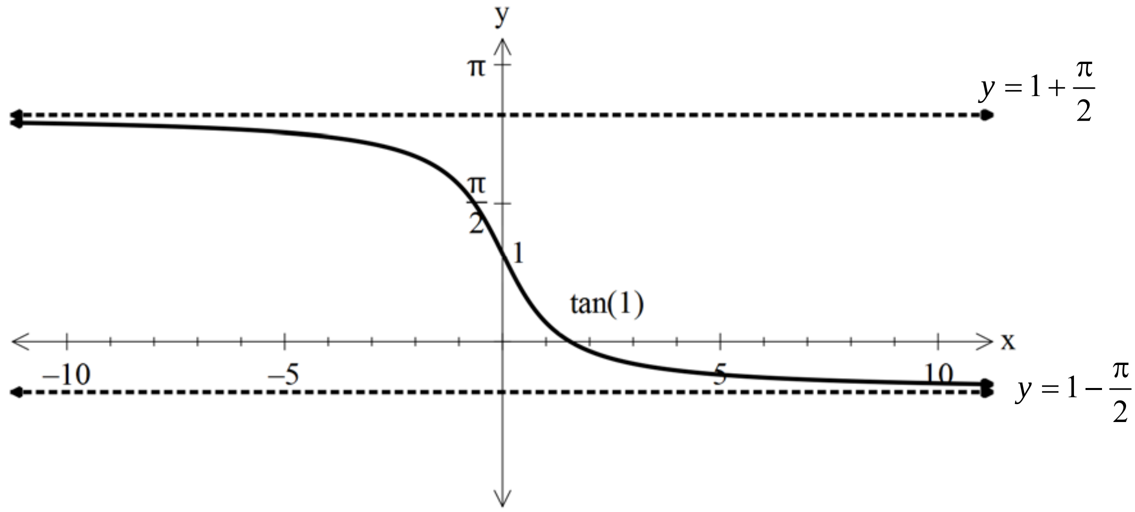
(b)	(i)	Write down the x-intercept of $y = 1 - \tan^{-1} x$.	**1**

For x-intercept, set $y = 0$

$1 - \tan^{-1} x = 0 \Rightarrow \tan^{-1} x = 1$

$x = \tan 1$

(ii)	Sketch the graph of $y = 1 - \tan^{-1} x$ showing all important features.	**2**



i) This question was completed quite poorly. Many were able to arrive at the fact that $1 = \tan^{-1} x$ but then decided that $x = \dfrac{\pi}{4}$

ii) This was completed well but the following were the common errors.

- The shape of the graph was not symmetrical at the y-intercept.
- The asymptotes were either not changed to $1 + \frac{\pi}{2}$ and $1 - \frac{\pi}{2}$.
- The asymptotes were correctly labelled as $1 + \frac{\pi}{2}$ and $1 - \frac{\pi}{2}$ but drawn equally distant

(c)	Find the general solutions of $\cos 3x - \cos x = 0$.	**2**

$\cos 3x = \cos x$

$3x = x + 2k\pi, -x + 2k\pi, \quad k \in Z$

$2x = 2k\pi \quad \text{or} \quad 4x = 2k\pi$

$x = k\pi \quad \text{or} \quad x = \dfrac{k\pi}{2}$

Most students did not use the method in the provided solutions, which is the easiest and most efficient method. Most students split $\cos 3x$ into $\cos(2x + x)$ and then proceeded to find an equation in just $\cos x$. If students were able to do this without making any errors then they mostly able to find the general solution

(d) Consider the function $f(x) = 2 + \sqrt{x}$.

(i) State the domain and range of $f(x)$. **1**

Domain : $x \geq 0$
Range: $y \geq 2$

(ii) Derive the equation of $f^{-1}(x)$. **1**

For the inverse,

$$x = 2 + \sqrt{y}$$
$$\sqrt{y} = x - 2$$
$$y = (x-2)^2, \quad x \geq 2$$

(iii) Find any points of intersection of $f(x)$ and $f^{-1}(x)$. **2**

Points of intersection occur on $y = x$. Solving $f^{-1}(x) = x$:

$$(x-2)^2 = x$$
$$x^2 - 4x + 4 = x$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$
$$x = 1, 4$$

But $x \geq 2$, therefore the only point of intersection is at $(4,4)$.

i)	Completed very well
ii)	Students were able to find the inverse on $f(x)$ but most did not consider that the domain needed to be restricted to $x \geq 2$. (No mark deduction was applied for not considering the domain.)
iii)	Most students realised that the functions intersect on the line $y = x$, however many neglected to realise that the point (1,1) was not in the original function. Some students attempted to solve the two functions simultaneously and were then unable to do so.

Question 8 (11 marks) Use a SEPARATE writing booklet.

(a) (i) Given $a > 1$, use the substitution $x = \frac{1}{u}$ to rewrite the integral **2**

$\int_1^a \frac{dx}{1+x^2}$ as an integral in u.

$$\int_1^a \frac{dx}{1+x^2} = \int_1^{\frac{1}{a}} \frac{1}{1+\left(\frac{1}{u}\right)^2}\cdot\left(-\frac{1}{u^2}du\right)$$

$$= \int_1^{\frac{1}{a}} \frac{\cancel{u^2}}{u^2+1}\cdot\left(-\frac{1}{\cancel{u^2}}du\right)$$

$$= \int_{\frac{1}{a}}^{1} \frac{du}{u^2+1}$$

Let $x = \frac{1}{u}$

$\frac{dx}{du} = -\frac{1}{u^2}$

$dx = -\frac{1}{u^2}du$

When $x = 1, u = 1$

When $x = a, u = \frac{1}{a}$

(ii) Hence, deduce the value of $\tan^{-1}(a) + \tan^{-1}\left(\frac{1}{a}\right)$. **2**

From (i) $\int_1^a \frac{dx}{1+x^2} = \int_{\frac{1}{a}}^{1} \frac{du}{u^2+1}$

$$\tan^{-1} a - \tan^{-1} 1 = \tan^{-1} 1 - \tan^{-1}\left(\frac{1}{a}\right)$$

$$\tan^{-1} a + \tan^{-1}\left(\frac{1}{a}\right) = 2\tan^{-1} 1 = 2\left(\frac{\pi}{4}\right)$$

$$\tan^{-1} a + \tan^{-1}\left(\frac{1}{a}\right) = \frac{\pi}{2}$$

In part (i) you were only required to give an integral in terms of u. Several students evaluated the integral.

Part (ii) was a "hence" question, you had to use the result in part (i). Using other methods to arrive at the result could not be awarded any marks.

(b) Consider the curve $y = \log_e x$ for $x \geq 1$. A variable point P lies on the curve and the line joining P to the origin makes an angle of θ with the positive x-axis.

It can be shown that $\theta = \tan^{-1}\left(\frac{\log_e x}{x}\right)$.

(i) Show that $\frac{d\theta}{dx} = \frac{1 - \log_e x}{x^2 + (\log_e x)^2}$. **1**

$$\frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{\log_e x}{x}\right)^2} \times \frac{\frac{1}{x}.x - \log_e x.1}{x^2}$$

$$= \frac{\cancel{x^2}}{x^2 + (\log_e x)^2} \times \frac{1 - \log_e x}{\cancel{x^2}}$$

$$= \frac{1 - \log_e x}{x^2 + (\log_e x)^2}$$

(ii) Hence, find the value of x for which θ is maximum. **2**

At stationary points, $\frac{d\theta}{dx} = 0$

$1 - \log_e x = 0$

$\log_e x = 1$

$x = e$

Testing nature of stationary point: the sign of $\frac{d\theta}{dx}$ is the same as the sign of $1 - \log_e x$ as $x^2 + (\log_e x)^2 > 0$

x	$\frac{1}{e}$	e	e^2
$1 - \log_e x$	2	0	-1
	+	0	–

$\therefore$ maximum at $x = e$

(iii) Explain the geometrical significance of this result. **1**

When $x = e$, OP is a tangent to the curve $y = \log_e x$.

At $x = e$, the gradient of OP is $\tan\theta = \dfrac{\log_e e}{e} = \dfrac{1}{e}$

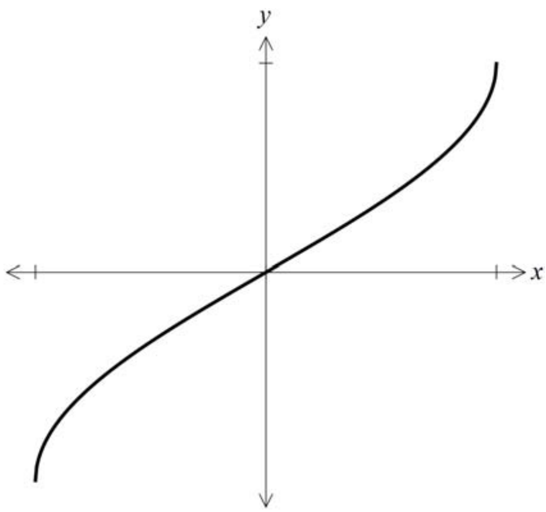
This is also the gradient of the curve $y = \log_e x$ at $x = e$.

Part (i) is a "show that". Show all steps of working so you can be awarded marks. When using the quotient rule together with chain rule try to break down the working into manageable chunks. For instance, $\dfrac{d}{dx}\left(\dfrac{\log x}{x}\right)$.

Students are encouraged to show values when testing for the nature of the stationary point especially when the question tells you if it is a maximum or minimum.

In part (iii), you needed to say the line was a tangent to the curve at this point to earn the mark. Simply restating that θ is maximum at this point was not sufficient.

(c) Consider the graph of the function $f(x) = \sin^{-1}\left(\dfrac{x}{a}\right)$ shown below. **3**



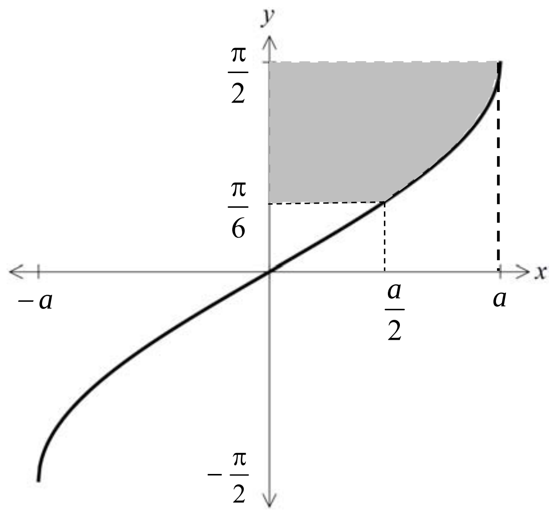
Without finding the primitive of $\sin^{-1}\left(\dfrac{x}{a}\right)$, evaluate $\displaystyle\int_{-\frac{a}{2}}^{a} \sin^{-1}\left(\dfrac{x}{a}\right)dx$

as an exact value in terms of a.

$$\int_{-\frac{a}{2}}^{a} \sin^{-1}\left(\frac{x}{a}\right)dx = \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin^{-1}\left(\frac{x}{a}\right)dx + \int_{\frac{a}{2}}^{a} \sin^{-1}\left(\frac{x}{a}\right)dx$$

$$= \underset{\text{odd function}}{0} + \int_{\frac{a}{2}}^{a} \sin^{-1}\left(\frac{x}{a}\right)dx$$

This is the area under the curve $y = \sin^{-1}\left(\frac{x}{a}\right)$ from $\frac{a}{2}$ to a. We evaluate this by finding the area between the curve and the y-axis from $\frac{\pi}{6}$ to $\frac{\pi}{2}$ and subtracting it from a suitable rectilinear area.



$y = \sin^{-1}\left(\frac{x}{a}\right) \Rightarrow x = a\sin y$.

$$\int_{\frac{a}{2}}^{a} \sin^{-1}\left(\frac{x}{a}\right)dx = \left[\frac{\pi}{2}\times a - \frac{\pi}{6}\times\frac{a}{2}\right] - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a\sin y\,dy$$

$$= \frac{5\pi a}{12} + \Big[a\cos y\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{5\pi a}{12} + \left[0 - a.\frac{\sqrt{3}}{2}\right]$$

$$= \frac{a\left(5\pi - 6\sqrt{3}\right)}{12}$$

Students are encouraged to draw a diagram and mark relevant areas on the graph. Students who used symmetry to simplify the required integral to either $\int_{\frac{a}{2}}^{a} \sin^{-1}\left(\frac{x}{a}\right)dx$ or $\int_{0}^{a} \sin^{-1}\left(\frac{x}{a}\right)dx - \int_{0}^{\frac{a}{2}} \sin^{-1}\left(\frac{x}{a}\right)dx$ were more likely to be successful as the areas involved are all in the first quadrant.

Common mistakes were in using areas to find integrals but failing to account for the fact that integrals are signed areas ie areas below the axis are treated as negative, combining areas incorrectly and algebraic errors.